Groupoid and Inverse Semigroup Presentations of Ultragraph C^*-Algebras

Alberto E. Marrero* and Paul S. Muhly†

Department of Mathematics-Physics
University of Puerto Rico
Cayey, PR 00736

Department of Mathematics
The University of Iowa
Iowa City, IA 52242

alberto_marrero@yahoo.com
pmuhly@math.uiowa.edu

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Abstract

Inspired by the work of Paterson on C^*-algebras of directed graphs, we show how to associate a groupoid $\mathfrak{G}_{\mathcal{G}}$ to an ultragraph $\mathcal{G}$ in such a way that the C^*-algebra of $\mathfrak{G}_{\mathcal{G}}$ is canonically isomorphic to Tomforde's C^*-algebra $C^*(\mathcal{G})$. The groupoid $\mathfrak{G}_{\mathcal{G}}$ is built from an inverse semigroup $S_{\mathcal{G}}$ naturally associated to $\mathcal{G}$.

1 INTRODUCTION

Cuntz and Krieger described a way to associate a C^*-algebra $\mathcal{O}_A$ to a finite square matrix A with entries in $\{0,1\}$ in [1]. Subsequently, their work was set in the context of graphs by a number of authors (see, e.g. [12]). It was soon recognized that infinite graphs led to problems that were not covered in [1] and numerous attempts over the years have been advanced for dealing with them. One very interesting attempt was introduced by Mark Tomforde in [10], where he defined the notion of an *ultragraph*. Roughly speaking, an ultragraph is a

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generalization of directed graph in which the range of an edge is allowed to be a *set* of vertices rather than just a single vertex. In [10] and [11], Tomforde showed that the class of ultragraph C^*-algebras includes all graph C^*-algebras and all so-called Exel-Laca algebras, as well as C^*-algebras that are in neither of these classes. Our goal in this note is to determine a groupoid model $\mathfrak{G}_{\mathcal{G}}$ for an ultragraph $\mathcal{G}$ in such a way that $C^*(\mathcal{G}) \simeq C^*(\mathfrak{G}_{\mathcal{G}})$, revealing salient features of $C^*(\mathcal{G})$. The groupoid connection enables one to interpret properties of $C^*(\mathcal{G})$ in dynamical terms. In the directed graph setting, this has been done with considerable consequence in [5] and [7].

Our approach is inspired by Paterson's paper [7]. We first build an inverse semigroup $S_{\mathcal{G}}$ that is designed to reflect the representation theory of $\mathcal{G}$. A representation of $\mathcal{G}$ is determined by certain partial isometries on a Hilbert space indexed by vertices and edges from $\mathcal{G}$. The C^*-algebra $C^*(\mathcal{G})$ of $\mathcal{G}$ is the universal C^*-algebra for such representations [10, Theorem 2.11]. A representation of $\mathcal{G}$ may be viewed directly as a representation of $S_{\mathcal{G}}$ by partial isometries. The groupoid model $\mathfrak{G}_{\mathcal{G}}$ we construct is built from $S_{\mathcal{G}}$ based on the approach developed by Paterson and exposed in his book [6]. We first build the universal groupoid canonically associated to $S_{\mathcal{G}}$ and then take a certain reduction for $\mathfrak{G}_{\mathcal{G}}$. We will call $\mathfrak{G}_{\mathcal{G}}$ the *ultrapath groupoid of* $\mathcal{G}$.

For a bit more detail to help with the motivation, recall that if E is an ordinary directed graph (but not necessarily row finite or without sinks), Paterson's inverse semigroup S_E, is the set of all pairs (α, β), where α, β are finite paths in the graph E and $r(\alpha) = r(\beta)$, together with a zero element z [7]. The multiplication in S_E is defined as follows: $(\alpha, \alpha'\mu)(\alpha', \beta) := (\alpha, \beta\mu)$, $(\alpha, \alpha')(\alpha'\mu, \beta') := (\alpha\mu, \beta')$ and all other products are the zero z. The involution on S_E is transposition: $(\alpha, \beta)^* := (\beta, \alpha)$. In an ordinary graph the paths of length zero are just the vertices. However in the ultragraph case, the paths of length zero are the *sets* in a space that we denote by $\mathcal{G}^0$, which is defined to be the smallest subcollection of subsets of G^0, that contains $\{v\}$, for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite union and intersections. That is, $\mathcal{G}^0$ is the "lattice" generated by $\{\{v\} \mid v \in G^0\}$ and the sets $r(e)$, $e \in \mathcal{G}^1$. We place "lattice" in double quotes because $\mathcal{G}^0$ may fail to be a lattice in the usual sense in that it may fail to contain all of G^0. Roughly speaking, we think of enlarging G^0 by adding in additional vertices, one for each *set* $r(e)$, $e \in \mathcal{G}^1$, so that the elements of $\{v : v \in G^0\} \cup \{r(e) : e \in \mathcal{G}^1\}$ play the role of "generalized vertices". Then the "lattice" $\mathcal{G}^0$ plays the role of "subsets of generalized vertices". Thus, since $C^*(\mathcal{G})$ involves partial isometries indexed by those special sets, we introduce the set $\mathfrak{p}$ consisting of $\mathcal{G}^0$ together with the set of all pairs (α, A), where α is a finite path in $\mathcal{G}$ with positive length, and $A \in \mathcal{G}^0$, with $A \subseteq r(\alpha)$. That is, $\alpha = \alpha_1\alpha_2 \cdots \alpha_l$ with $s(\alpha_{i+1}) \in r(\alpha_i)$. We will call $\mathfrak{p}$ the *ultrapath space* of the ultragraph $\mathcal{G}$. The range map r and the source map s extend to $\mathfrak{p}$ in a natural way. Our inverse semigroup $S_{\mathcal{G}}$ is the set of pairs $(x, y) \in \mathfrak{p} \times \mathfrak{p}$ such that $r(x) = r(y)$. The operations on $S_{\mathcal{G}}$ are defined similarly to those on S_E. However, the structure of $S_{\mathcal{G}}$ is rather more complicated and much of our analysis is devoted to keeping track of the complications.

The next section is devoted to the constructions of $S_{\mathcal{G}}$ and $\mathfrak{G}_{\mathcal{G}}$ and to showing

that $C^*(\mathcal{G}) \simeq C^*(\mathfrak{G}_\mathcal{G})$. In the subsequent section we address the amenability of $C^*(\mathcal{G})$. We use a groupoid crossed product argument to show that $C^*(\mathfrak{G}_\mathcal{G})$ is nuclear and hence that $C^*(\mathfrak{G}_\mathcal{G})$ and $C^*(\mathcal{G})$ are amenable. The last section deals with the *simplicity* of $C^*(\mathcal{G})$. We define an analogue of the so-called "condition (K)" that appears in the analysis of ordinary graph C^*-algebras. We show that $\mathfrak{G}_\mathcal{G}$ is essentially principal if and only if $\mathcal{G}$ satisfies condition (K). When $\mathcal{G}$ satisfies condition (K), then thanks to the amenability of $\mathfrak{G}_\mathcal{G}$, the (norm closed, two sided) ideals in $C^*(\mathcal{G})$ and $C^*(\mathfrak{G}_\mathcal{G})$ are parameterized by the open invariant subsets of the unit space of $\mathfrak{G}_\mathcal{G}$. In particular, $C^*(\mathcal{G})$ and $C^*(\mathfrak{G}_\mathcal{G})$ are simple if and only if $\mathfrak{G}_\mathcal{G}$ is minimal.

1.1 Notation and Conventions

We set up here the basic notation we shall use for graphs, ultragraphs and inverse semigroups. Additional notation will be developed as needed, below.

A directed graph $E = \left(E^0, E^1, r, s\right)$ consists of a countable set of vertices E^0, a countable set of edges E^1, and maps r, $s : E^1 \to E^0$ identifying the range and source of each edge. The graph E is called *row finite* if for each $v \in E^0$, the set of edges starting at v is finite. The graph is called *locally finite* if for each vertex $v \in E^0$, the set of edges starting at v is finite and the set of edges terminating at v is also finite. A vertex v is called a *sink*, if there no edges starting at v. A *finite path* is a sequence α of edges $e_1 \dots e_k$ where $s(e_{i+1}) = r(e_i)$ for $1 \leq i \leq k-1$. We write $\alpha = e_1 \dots e_k$. The length $l(\alpha)$ of α is just k. Each vertex v is regarded as a finite path of length zero. The source and range maps extend to the set of finite paths in the natural way. The set of infinite paths, is the set of infinite sequences of edges $\gamma = e_1 e_2 \dots$, such that $s(e_{i+1}) = r(e_i)$ for all i. The source map extends to that set in the natural way as well.

Following [10], an *ultragraph* is a system $\mathcal{G}= \left(G^0, \mathcal{G}^1, r, s\right)$, where G^0 and $\mathcal{G}^1$ are countable sets, called, respectively, the *vertices* and *edges* of $\mathcal{G}$; where s is a function from $\mathcal{G}^1$ to G^0, called the *source* function; and where r is a function from $\mathcal{G}^1$ to the power set of G^0, $\mathcal{P}\left(G^0\right)$, such that $r(e)$ is non-empty for each $e \in \mathcal{G}^1$. We write $\mathcal{G}^0$ for the smallest subcollection of $\mathcal{P}\left(G^0\right)$ that contains $\{v\}$, for each $v \in G^0$ and contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite union and intersections. A *finite path* in $\mathcal{G}$ is either an element of $\mathcal{G}^0$ or a sequence of edges $e_1 \dots e_k$ in $\mathcal{G}^1$ where $s(e_{i+1}) \in r(e_i)$ for $1 \leq i \leq k$. If we write $\alpha = e_1 \dots e_k$, the length $|\alpha|$ of α is just k. The length $|A|$ of a path $A \in \mathcal{G}^0$ is zero. We define $r(\alpha) = r(e_k)$ and $s(\alpha) = s(e_1)$. For $A \in \mathcal{G}^0$, we set $r(A) = A = s(A)$. The set of finite paths in $\mathcal{G}$ is denoted by $\mathcal{G}^*$. The set of infinite paths $\gamma = e_1 e_2 \dots$ in $\mathcal{G}$ is denoted by $\mathfrak{p}^\infty$. The length $|\gamma|$ of $\gamma \in \mathfrak{p}^\infty$ is defined to be ∞. A vertex v in $\mathcal{G}$ is called a *sink* if $\left|s^{-1}(v)\right| = 0$ and is called an *infinite emitter* if $\left|s^{-1}(v)\right| = \infty$. We say that a vertex v is a *singular vertex* if it is either a sink or an infinite emitter. Finally given vertices $v, w \in G^0$, we write $w \geq v$ to mean that there exists a path $\alpha \in \mathcal{G}^*$ with $s(\alpha) = w$ and $v \in r(\alpha)$. Also we write $G^0 \geq \{v\}$ to mean that $w \geq v$, for all $w \in G^0$. See [11, p.8].

Blanket Assumption 1 Throughout the paper we will assume that there are no sinks in $\mathcal{G}$, unless otherwise specified.

The reason for this is that we want to investigate $C^*(\mathcal{G})$ using an ultragraph groupoid $\mathfrak{G}_{\mathcal{G}}$, whose unit space $\mathfrak{G}_{\mathcal{G}}^{(0)}$ consists of paths that cannot end at a sink. So if we want to examine $C^*(\mathcal{G})$ from a groupoid perspective, then sinks must be excluded.

A semigroup S is called an *inverse semigroup* if for each $s \in S$, there exists a unique element $t \in S$ such that $sts = s$ and $tst = t$. We write the element t as s^*. Note that $s^{**} = s$. Every element ss^* belongs to the set $E(S)$ of *idempotents* of S. The set $E(S)$ is a commutative subsemigroup of S and so is a semilattice. There is a natural order on $E(S)$ given by declaring $e \leq f$ if and only if $ef = e$, e and $f \in E(S)$, see [6, Proposition 2.1.1, p.22].

2 THE INVERSE SEMIGROUP $S_{\mathcal{G}}$ OF AN ULTRAGRAPH $\mathcal{G}$ AND THE UNIVERSAL GROUPOID FOR $S_{\mathcal{G}}$

In this section we have two main objectives. The first is to obtain an inverse semigroup model $S_{\mathcal{G}}$ for an ultragraph $\mathcal{G}$. The second is to identify the universal groupoid $H_{\mathcal{G}}$ for $S_{\mathcal{G}}$. The definition of $S_{\mathcal{G}}$ stems from the representation theory of $\mathcal{G}$. Recall the following definition due to Tomforde [10].

Definition 2 *A representation of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$ is given by a family $\{p_A : A \in \mathcal{G}^0\}$ of projections, and a family $\{s_e : e \in \mathcal{G}^1\}$ of partial isometries with mutually orthogonal ranges such that:*

(i) $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$, *and* $p_{A\cup B} = p_A + p_B - p_{A\cap B}$, *for all* $A, B \in \mathcal{G}^0$*;*

(ii) $s_e^* s_e = p_{r(e)}$, *for all* $e \in \mathcal{G}^1$*;*

(iii) $s_e s_e^* \leq p_{s(e)}$, *for all* $e \in \mathcal{G}^1$*;*

(iv) $p_v = \sum_{s(e)=v} s_e s_e^*$, *whenever* $0 < \left|s^{-1}(v)\right| < \infty$.

The family $\{s_e, p_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\}$ is also called a *Cuntz-Krieger* $\mathcal{G}$-family. For a path $\alpha := e_1 \dots e_n \in \mathcal{G}^*$ we define s_α to be $s_{e_1} \cdots s_{e_n}$ if $|\alpha| \geq 1$ and p_A if $\alpha = A \in \mathcal{G}^0$.

Every inverse semigroup can be realized as *-semigroup of partial isometries on a Hilbert space. See [6, Proposition 2.1.4]. If $\mathcal{G} = \left(G^0, \mathcal{G}^1, r, s\right)$ is an ultragraph, and if $\{s_e, p_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\}$ is a universal Cuntz-Krieger $\mathcal{G}$-family realized on a Hilbert space $\mathcal{H}$, we know that the C^*-algebra $C^*(\mathcal{G})$ may be identified with the closed span, $\overline{span}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{G}^*, A \in \mathcal{G}^0\}$, see [10, p.7]. Note that for each α, $\beta \in \mathcal{G}^*$, and $A \in \mathcal{G}^0$ with $A \subseteq r(\alpha) \cap r(\beta)$, the operator on $\mathcal{H}$, $T_{((\alpha,A),(\beta,A))} := s_\alpha p_A s_\beta^*$ is a partial isometry, such that $T^*_{((\alpha,A),(\beta,A))} = T_{((\beta,A),(\alpha,A))}$. So we obtain a *-semigroup of partial isometries on $\mathcal{H}$, and therefore an inverse semigroup, which we shall denote by $S_{\mathcal{G}}$. Using

properties of the generators for $C^*(\mathcal{G})$, $s_\alpha p_A s_\beta^*$, α, $\beta \in \mathcal{G}^*$, and $A \in \mathcal{G}^0$ (see [10, Lemma 2.8 and Lemma 2.9]), we will describe the inverse semigroup $S_\mathcal{G}$ for $\mathcal{G}$ in a fashion that is independent of any Hilbert space representation. It turns out that our inverse semigroup is analogous to the inverse semigroup model for an ordinary graph that Paterson obtains in [7].

Remark 3 *While it is clear in a broad sense what one must do to follow the path laid out by Paterson, there is an important difficulty that must be surmounted. It may be helpful, therefore, to call attention to it here for the purpose of motivating later detail. Condition (i) in Definition 2 is the source of the difficulty. Notice that it says that the family $\{p_A : A \in \mathcal{G}^0\}$ is a "proto-spectral measure" defined on the "lattice" $\mathcal{G}^0$. As we build our inverse semigroup and groupoid models, we will have to keep track of how to guarantee condition (i) in what we are doing.*

For this purpose, we find it helpful to "enrich" the path notation discussed in subsection 1.1 and introduce the notion of what we like to call "ultrapaths". For $n \geq 1$, we define $\mathfrak{p}^n := \{(\alpha, A) : \alpha \in \mathcal{G}^*, |\alpha| = n, A \in \mathcal{G}^0, A \subseteq r(\alpha)\}$. We specify that $(\alpha, A) = (\beta, B)$ if and only if $\alpha = \beta$ and $A = B$. We set $\mathfrak{p}^0 := \mathcal{G}^0$ and we let $\mathfrak{p} := \coprod_{n\geq 0} \mathfrak{p}^n$. We define the length of a pair (α, A), $|(\alpha, A)|$ to be the length of α, $|\alpha|$. We call $\mathfrak{p}$ the *ultrapath space* associated with $\mathcal{G}$ and the elements of $\mathfrak{p}$ are called *ultrapaths*. We may extend the range map r and the source map s to $\mathfrak{p}$ by the formulas, $r((\alpha, A)) = A$ and $s((\alpha, A)) = s(\alpha)$. Each $A \in \mathcal{G}^0$ is regarded as an ultrapath of length zero and we define $r(A) = s(A) = A$. It will be convenient to embed $\mathcal{G}^*$ in $\mathfrak{p}$ by sending α to $(\alpha, r(\alpha))$, if $|\alpha| \geq 1$, and by sending A to A for all $A \in \mathcal{G}^0$ (See [10].) In a sense, our formation of the ultrapath space $\mathfrak{p}$ is analogous to the process of forming the *disjoint union* of a family of not-necessarily-disjoint sets, i.e., their co-product.

Notation 4 *Generic elements of $\mathfrak{p}$ will be denoted by lower case letters at the end of the alphabet: x, y, z and w. However if $x \in \mathfrak{p}$, and if $x \in \mathfrak{p}^0 := \mathcal{G}^0$, we think of x as a set A in $\mathcal{G}^0$. Otherwise we think of x as a pair, say (α, A), with $|\alpha| \geq 1$. Algebraically, we treat $\mathfrak{p}$ like a small category and say that a product $x \cdot y$ is defined only when $r(x) \cap s(y) \neq \emptyset$*[1]*. When $x \cdot y$ is defined, the product is effectively concatenation of x and y. That is, if $x = (\alpha, A)$ and $y = (\beta, B)$, then $x \cdot y$ is defined if and only if $s(\beta) \in A$, and in this case, $x \cdot y := (\alpha\beta, B)$. Also we specify that:*

$$x \cdot y = \begin{cases} x \cap y & \text{if } x, y \in \mathcal{G}^0 \text{ and if } x \cap y \neq \emptyset \\ y & \text{if } x \in \mathcal{G}^0,\ |y| \geq 1, \text{ and if } x \cap s(y) \neq \emptyset \\ x_y & \text{if } y \in \mathcal{G}^0,\ |x| \geq 1, \text{ and if } r(x) \cap y \neq \emptyset \end{cases} \tag{1}$$

where, if $x = (\alpha, A)$, $|\alpha| \geq 1$ and if $y \in \mathcal{G}^0$, the expression x_y is defined to be $(\alpha, A \cap y)$. Observe also that the range of x_y, $r(x_y)$, becomes $r(x_y) = r(x) \cap y$.

[1]The notation is a little inconsistant. By definition, $s(y)$ is a set only when $y \in \mathfrak{p}^0$; otherwise, $s(y)$ is a point in G^0. In the latter case, we really want to identify $s(y)$ with $\{s(y)\}$. We shall do this whenever it is convenient and not add extra notation to distinguish between $s(y)$ and $\{s(y)\}$.

Given $x, y \in \mathfrak{p}$, we say that x has y as an initial segment if $x = y \cdot x'$, for some $x' \in \mathfrak{p}$, with $s(x') \cap r(y) \neq \emptyset$. We shall say that x and y in $\mathfrak{p}$ are comparable if x has y as an initial segment or vice versa. Furthermore, the equation, $x = x \cdot y$, holds if and only if $y \in \mathcal{G}^0$ and $r(x) \subseteq y$.

Recall from subsection 1.1 that $\mathfrak{p}^\infty$ denotes the set of all infinite paths. We extend the source map s to $\mathfrak{p}^\infty$, by defining $s(\gamma) = s(e_1)$, where $\gamma = e_1 e_2 \ldots$. We may concatenate pairs in $\mathfrak{p}$, with infinite paths in $\mathfrak{p}^\infty$ as follows. If $y = (\alpha, A) \in \mathfrak{p}$, and if $\gamma = e_1 e_2 \ldots \in \mathfrak{p}^\infty$ are such that $s(\gamma) \in r(y) = A$, then the expression $y \cdot \gamma$ is defined to be $\alpha\gamma = \alpha e_1 e_2 ... \in \mathfrak{p}^\infty$. If $y = A \in \mathcal{G}^0$, we define $y \cdot \gamma = A \cdot \gamma = \gamma$ whenever $s(\gamma) \in A$. Of course $y \cdot \gamma$ is not defined if $s(\gamma) \notin r(y) = A$. In this way, we get an "action" of $\mathfrak{p}$ on $\mathfrak{p}^\infty$.

Definition 5 *Let $S_{\mathcal{G}} := \{(x, y) : x, y \in \mathfrak{p}, r(x) = r(y)\} \cup \{\omega\}$. We define an involution on $S_{\mathcal{G}}$ by $\omega^* = \omega$ and $(x, y)^* = (y, x)$, and we define a product on $S_{\mathcal{G}}$ by the following requirements:*

1. $(x, r(x))(r(y), y) := (x \cdot r(y), y \cdot r(x))$, for all x and $y \in \mathfrak{p}$ with $r(x) \cap r(y) \neq \emptyset$.
2. If x has z as an initial segment, so $x = z \cdot x'$ for some $x' \in \mathfrak{p}$, then $(w, z)(x, y) = (w, z)(z \cdot x', y) := (w \cdot x', y)$.
3. If z has x as an initial segment, so $z = x \cdot z'$ for some $z' \in \mathfrak{p}$, then $(w, z)(x, y) = (w, x \cdot z')(x, y) := (w, y \cdot z')$.
4. All other products are defined to be ω.

Proposition 6 *The set $S_{\mathcal{G}}$ with this involution and product is an inverse semigroup.*

Proof. We first have to show that the product in $S_{\mathcal{G}}$ is associative. If one of the terms is ω, then this is obvious. So let $(x_i, y_i) \in S_{\mathcal{G}}$ $(1 \leq i \leq 3)$, and set $s_1 = (x_1, y_1)((x_2, y_2)(x_3, y_3))$ and $s_2 = ((x_1, y_1)(x_2, y_2))(x_3, y_3)$. We have to show that $s_1 = s_2$.

The cases that give $s_1 \neq \omega$ are the following (for appropriate ultrapaths z, w):

1. $y_2 = x_3 \cdot z$, for some $z \in \mathfrak{p}$ and $y_1 = x_2 \cdot w$ for some $w \in \mathfrak{p}$;
2. $y_2 = x_3$, and $y_1 = x_2 \cdot w$ for some $w \in \mathfrak{p}$;
3. $y_2 = x_3 \cdot z$, for some $z \in \mathfrak{p}$ and $y_1 = x_2$;
4. $y_2 = x_3 \cdot z$, for some $z \in \mathfrak{p}$ and $x_2 = y_1 \cdot w$ for some $w \in \mathfrak{p}$;
5. $x_i \in \mathcal{G}^0$, for some i, with $s(x_i) \cap r(y_{i-1}) \neq \emptyset$, together with the cases above;

6. $y_i \in \mathcal{G}^0$, for some i, with $r(x_{i+1}) \cap s(y_i) \neq \emptyset$, together with the cases above.

One checks directly that in each case, $s_2 = s_1$. (In case 4, one needs to consider separately the cases $x_2 = y_1 \cdot z'$ and $y_1 = x_2 \cdot z'$). So if $s_1 \neq \omega$, then $s_1 = s_2$. Similarly, one shows that if $s_2 \neq \omega$, then $s_2 = s_1$. The associative law then follows.

Next we have to show that for each $s \in S_{\mathcal{G}}$, s^* is the only s' for which $ss's = s$ and $s'ss' = s'$. If $s = \omega$, then this is trivial. So let $s = (x, y) \neq \omega$. If s' is such that $ss's = s$, then $s' = (w, z)$ for some ultrapaths w, z, with z, x and y, w comparable. (See Notation 1.1) Suppose first z has x as an initial segment. We shall show that $z = x$. A similar argument will show that $x = z$ if x has z as an initial segment. So if z has x as an initial segment then $z = x \cdot z'$ for some $z' \in \mathfrak{p}$ with $s(z') \cap r(x) \neq \emptyset$. Then from the equation $ss's = s$ and from the definition of the product on $S_{\mathcal{G}}$, the equality, $(x, y) = (x, y)(w, y \cdot z')$, holds. We see then that the product, $(x, y)(w, y \cdot z')$, is not ω. So we have to consider two cases.

Case I w has y as an initial segment. So $w = y \cdot w'$ for some $w' \in \mathfrak{p}$ with $s(w') \cap r(y) \neq \emptyset$. Then we have $(x, y) = (x, y)(y \cdot w', y \cdot z') = (x \cdot w', y \cdot z')$, by definition. (See 2 Definition 5) Thus the equation, $(x \cdot w', y \cdot z') = (x, y)$, holds, which implies that the equation, $y \cdot z' = y$, holds as well. Therefore z' belongs to $\mathcal{G}^0$, and $r(x) = r(y) \subseteq z'$. Then $x = x \cdot z' = z$.

Case II y has w as an initial segment. In this case a similar proof gives us that $x = z$ as well.

Similarly, by considering the equation $s'ss' = s'$, one can show that $w = y$, in the situation when y and w are comparable. Then $s' = s^*$. Clearly $ss^*s = s$ and $s^*ss^* = s^*$. ■

The next theorem shows that there is a bijection between the class of representations of the ultragraph $\mathcal{G}$, and *certain* class of representations of the inverse semigroup $S_{\mathcal{G}}$. Compare with [7, Theorem 2, (a), (b)], and of course please keep in mind Remark 3.

Theorem 7 *There is a natural one-to-one correspondence between :*

(a) the class $R_{\mathcal{G}}$ of representations of $\mathcal{G}$; and

(b) the class $R_{S_{\mathcal{G}}}$ of representations π of $S_{\mathcal{G}}$ such that:

(i) $\pi(\omega) = 0$,

(ii) $\pi(v, v) - \sum_{s(e)=v} \pi(e, e) = 0$, *for every* $v \in G^0$, *with* $0 < |s^{-1}(v)| < \infty$, *and*

(iii) $\pi(A \cup B, A \cup B) - \pi(A, A) - \pi(B, B) + \pi(A \cap B, A \cap B) = 0$, *for every* A, $B \in \mathcal{G}^0$.

Proof. Let $\{p_A, s_e : A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$ be a representation of $\mathcal{G}$, realized on a Hilbert Space $\mathcal{H}$. Define a $*$-map $\pi : S_{\mathcal{G}} \to B(\mathcal{H})$ by $\pi(x,y) = S_x S_y^*$, and $\pi(\omega) = 0$, where S_x is defined to be $s_\alpha p_A$ if $x = (\alpha, A)$, and p_A if $x = A \in \mathcal{G}^0$. Then π is a $*$-homomorphism. The proof is simple. One checks that $\pi(st) = \pi(s)\pi(t)$ for the different kinds of product using properties of the generator $s_\alpha p_A s_\beta^*$ for $C^*(\mathcal{G})$. See [10, Lemma 2.8 and Lemma 2.9]. For example checking carefully all the cases, we have $\pi[(w,z)]\pi[(z\cdot x', y)] = (S_w S_z^*)(S_{z\cdot x'} S_y^*) = S_{w\cdot x'} S_y^* {=} \pi[(w\cdot x', y)] = \pi[(w,z)(z\cdot x', y)]$. Since $\pi(\omega) = 0$ and $s_e s_e^* = \pi(e,e)$, it follows from (i) and (iv) of Definition 2 that $\pi \in R_{S_{\mathcal{G}}}$.

Conversely, any $\pi \in R_{S_{\mathcal{G}}}$ determines an element of $R_{\mathcal{G}}$ by taking $p_A =: \pi(A,A)$, if $A \neq \emptyset$ and 0 otherwise. For $e \in \mathcal{G}^1$ take $s_e := \pi(e, r(e))$. Since $p_\emptyset = 0$ and $p_A p_B = \pi(A,A)\pi(B,B) = \pi(A\cap B, A\cap B) = p_{A\cap B}$, (i) of Definition 2 follows. Also since $s_e s_e^* = \pi(e, r(e))\pi(r(e), e) = \pi(e,e)$, (iv) of Definition 2 follows as well. (i) and (iii) follow since π is a $*$-homomorphism on $S_{\mathcal{G}}$. This establishes the correspondence between the classes of representations of (a) and (b). ■

Every representation π of $S_{\mathcal{G}}$ by partial isometries on Hilbert space gives a bounded representation π of $l^1(S_{\mathcal{G}})$ in a natural way, and $C^*(S_{\mathcal{G}})$ is just the enveloping C^*-algebra of $l^1(S_{\mathcal{G}})$ obtained by taking the biggest norm coming from all such π's. The C^*-algebra that we want here, which we will denote by $C_0^*(S_{\mathcal{G}})$, is obtained in the same way but using only π's for which $\pi(\omega) = 0 = \pi((v,v) - \sum_{s(e)=v}(e,e))$, for every $v \in G^0$, with $0 < |s^{-1}(v)| < \infty$ and for which $\pi((A\cup B, A\cup B) - (A,A) - (B,B) + (A\cap B, A\cap B)) = 0$, for every A and $B \in \mathcal{G}^0$. In fact $C_0^*(S_{\mathcal{G}})$ is the quotient C^*-algebra $C^*(S_{\mathcal{G}})/I$, where I is the closed ideal of $C^*(l^1(S_{\mathcal{G}}))$ generated by elements of the form: ω, $(v,v) - \sum_{s(e)=v}(e,e)$, for every $v \in G^0$, with $0 < |s^{-1}(v)| < \infty$ and $(A\cup B, A\cup B) - (A,A) - (B,B) + (A\cap B, A\cap B)$, for every A and $B \in \mathcal{G}^0$. *A priori* the quotient $C_0^*(S_{\mathcal{G}})$ could be zero, but Tomforde shows that it isn't, in [10], and, of course, our analysis will show this, too.

The next objective is to identify the universal groupoid of $S_{\mathcal{G}}$ for a general ultragraph $\mathcal{G}$. The universal groupoid ([6, Ch. 4]) H of a countable inverse semigroup S is constructed as follows. The unit space $H^{(0)}$ of H is the set of non-zero semicharacters, i.e., homomorphisms, χ, from the commutative inverse subsemigroup of idempotents $E(S)$ in S to the semigroup $\{0,1\}$ (under multiplication). The topology on $H^{(0)}$ is the topology of pointwise convergence on $E(S)$. This implies that the family of sets $D_{e,e_1,\ldots,e_n} = D_e \cap D_{e_1}^c \cap \ldots \cap D_{e_n}^c$, (with c standing for "complement" and $e, e_i \in E(S)$, $e \geq e_i$, $1 \leq i \leq n$) is a basis for the topology of $H^{(0)}$, [6, Chap.4, p.174]. With respect to this topology, the space $H^{(0)}$ is locally compact, totally disconnected and Hausdorff, [6, p.173]. There is a natural right action of S on $H^{(0)}$ given as follows. First, an element $\chi \in H^{(0)}$ is in the domain D_s of $s \in S$ if $\chi(ss^*) = 1$. The element $\chi \cdot s \in H^{(0)}$ is then defined by the equation $(\chi \cdot s)(e) = \chi(ses^*)$, for $e \in E(S)$. The map $\chi \to \chi \cdot s$ is a homeomorphism from D_s onto D_{s^*}. Theorem 4.3.1 of [6] shows that the universal groupoid H for S is the quotient

$$\{(\chi, s) : \chi \in D_s, s \in S\} / \backsim$$

where, by definition, $(\chi, s) \backsim (\chi', t)$ whenever $\chi = \chi'$ and there exists $e \in E(S)$ such that $\chi(ss^*) = \chi(tt^*)$ and $es = et$. That is, two pairs (χ, s) and (χ', t) are equivalent if and only if $\chi = \chi'$ and s and t have the same germ at χ. The composable pairs are pairs of the form $\left(\overline{(\chi, s)}, \overline{(\chi \cdot s, t)}\right)$, where $\chi \in D_s$, $\chi \cdot s \in D_t$, and $s, t \in S$; and the product and inversion on H are given by the maps $\left(\overline{(\chi, s)}, \overline{(\chi \cdot s, t)}\right) \to \overline{(\chi, st)}$ and $\overline{(\chi, s)} \to \overline{(\chi \cdot s, s^*)}$, respectively. Also H is an r-discrete groupoid, where the topology on H is the germ topology. It has a basis consisting of sets of the form $D(U, s)$, where $s \in S$, U is an open subset of D_s, and $D(U, s) := \{\overline{(\chi, s)} : \chi \in U\}$. Further, the map Ψ, where $\Psi(s) = \{\overline{(\chi, s)} : \chi \in D_s\}$ is an inverse semigroup isomorphism from S into the ample semigroup H^a. (For any r-discrete groupoid G, the *ample semigroup* G^a is the inverse semigroup of compact open, Hausdorff G-sets in G, see [6, Chap.2, Definition 2.2.4, Proposition 2.2.6]).

The description of the universal groupoid $H = H_{\mathcal{G}}$ of $S = S_{\mathcal{G}}$ is in many respects similar to the description of the universal groupoid of the graph inverse semigroup associated to a graph. However, there are some important differences. To highlight them, we follow as closely as possible the discussion for the directed graph inverse semigroup obtained by Paterson in [7]. The key is to identify the unit space $H^{(0)}$ of H. The semigroup of idempotents of $S_{\mathcal{G}}$, which we denote by $E(S_{\mathcal{G}})$, is the set $\{(x, x) : x \in \mathfrak{p}\} \cup \{\omega\}$. Recall that for any inverse semigroup S, there is a natural order on the idempotent subsemigroup $E(S)$ defined by the prescription $e \leq f$ if and only if $ef = e$, $(e,\ f \in E(S))$. In our setting, the order on $E(S_{\mathcal{G}})$ may be described in terms of path length and set inclusion, as the following remark indicates. We leave the proof to the reader.

Remark 8 *If the product in $E(S_{\mathcal{G}})$, $(x, x)(z, z)$, is not ω then the inequality, $(z, z) \leq (x, x)$, holds if and only if either $|z| > |x|$ or, if $|z| = |x|$, then $r(z) \subseteq r(x)$.*

As is the case with any idempotent semigroup, the elements in $E(S_{\mathcal{G}})$ can themselves be regarded as semicharacters on $E(S_{\mathcal{G}})$, and for each element, there is a filter associated with it. That is, given $e \in E(S_{\mathcal{G}})$, then χ_e is the semicharacter of $E(S_{\mathcal{G}})$ defined by: $\chi_e(f) = 1$ if $f \geq e$ and is 0 otherwise. Its filter $\tilde{e}$, is the set of idempotents $\tilde{e} := \{f \in E(S_{\mathcal{G}}) : f \geq e\}$, see [6, p.173-174]. That is, $\tilde{e}$ is the principal filter determined by e. Furthermore, the set $\widetilde{E(S_{\mathcal{G}})} := \{\tilde{e} : e \in E(S_{\mathcal{G}})\}$ is dense in the set of all nonzero semicharacters of $E(S_{\mathcal{G}})$, which we shall denote by $\widehat{E(S_{\mathcal{G}})}$. (See [6, Proposition 4.3.1 p.174][2].) The collection of "subsets of generalized vertices" in the ultragraph $\mathcal{G}$, which, recall, is denoted $\mathcal{G}^0$ and is an idempotent inverse semigroup in its own right under intersection, may be viewed as an sub-inverse-semigroup of $E(S_{\mathcal{G}})$ via the map $A \to (A, A)$. Then every semicharacter on $E(S_{\mathcal{G}})$ restricts to one on $\mathcal{G}^0$. This leads to the inclusion,

[2]It is customary to denote the principal filter determined by idempotent e by $\overline{e}$, and then we might write $\overline{E(S)}$ for the collection of all such filters. However, in our setting, this leads to awkward statements like "$\overline{E(S)}$ is dense in X", which in turn would lead one to believe $\overline{E(S)} = X$.

$H_{\mathcal{G}}^{(0)} = \widehat{E(S_{\mathcal{G}})} \subseteq \widehat{\mathcal{G}^0}$, where $\widehat{\mathcal{G}^0}$ denotes the set of all non-zero semicharacters of $\mathcal{G}^0$. The topology on $\widehat{\mathcal{G}^0}$ is the topology of pointwise convergence on $\mathcal{G}^0$. Consequently, the family of compact open sets, $D(A, A) := \{\chi \in \widehat{\mathcal{G}^0} : \chi(A) = 1\}$, $A \in \mathcal{G}^0$, is a subbasis. It follows that the space, $\widehat{\mathcal{G}^0}$, is locally compact and Hausdorff as well. (See [6, Chapter 4, p.174].) We want to emphasize here that the space $\widehat{\mathcal{G}^0}$ is *huge*. Since it contains the discrete space G^0, $\widehat{\mathcal{G}^0}$ contains the Stone-Čech compactification βG^0 of G^0. Since $\mathcal{G}^0$ is an idempotent inverse semigroup, the elements in $\mathcal{G}^0$ can themselves be regarded as semicharacters of $\mathcal{G}^0$, and each element determines a principal filter. That is, given $A \in \mathcal{G}^0$, then χ_A is the semicharacter of $\mathcal{G}^0$ defined by: $\chi_A(B) = 1$ if $A \subseteq B$ and is 0 otherwise. Its filter $\tilde{A}$, is the set given then by: $\tilde{A} = \{B \in \mathcal{G}^0 : A \subseteq B\}$. Consequently, the set of all semicharacters of the form, χ_A, $A \in \mathcal{G}^0$, which we shall denote by, $\widetilde{\mathcal{G}^0}$, is dense in $\widehat{\mathcal{G}^0}$ [6, Proposition 4.3.1 p.174]. This fact plays an important role in our efforts to overcome the difficulties alluded to in Remark 3.

Now we proceed to identify the unit space $H_{\mathcal{G}}^{(0)} = \widehat{E(S_{\mathcal{G}})}$, of the universal groupoid $H = H_{\mathcal{G}}$ of $S = S_{\mathcal{G}}$ and its topology in more concrete terms. It is more convenient to discuss the space $\widehat{E(S_{\mathcal{G}})}$ in terms of filters rather than in terms of semicharacters. Recall that given a nonzero semicharater χ in $\widehat{E(S_{\mathcal{G}})}$ its filter, $\mathcal{A}_\chi$, is the set $\mathcal{A}_\chi := \{(x, x) \in E(S_{\mathcal{G}}) : \chi(x, x) = 1\}$. (See [6, p.173-174].) Each ultrapath, $y \in \mathfrak{p}$, defines a semicharacter of $E(S_{\mathcal{G}})$ via the equation,

$$y(x, x) = \begin{cases} 1 & \text{if } (x, x)(y, y) \neq \omega,\ |x| < |y| \text{ or; if } |x| = |y|, \text{ then } r(x) \supseteq r(y) \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Also an infinite path $\gamma \in \mathfrak{p}^\infty$ defines a semicharacter of $E(S_{\mathcal{G}})$ via the equation:

$$\gamma(x, x) = \begin{cases} 1 & \text{if } \gamma = x \cdot \gamma', \gamma' \in \mathfrak{p}^\infty, s(\gamma') \in r(x) \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Consequently, the filter in $E(S_{\mathcal{G}})$ determined by the ultrapath y, $\mathcal{A}_y$, is the set, $\mathcal{A}_y := \{(x, x) \in E(S_{\mathcal{G}}) : (x, x)(y, y) \neq \omega\ ;\ |x| < |y| \text{ or, if } |x| = |y|, \text{ then } r(x) \supseteq r(y)\ \}$, while the filter in $E(S_{\mathcal{G}})$ determined by an infinite path γ, $\mathcal{A}_\gamma$, is the set $\mathcal{A}_\gamma := \{(x, x) \in E(S_{\mathcal{G}}) : \gamma = x \cdot \gamma', \gamma' \in \mathfrak{p}^\infty, s(\gamma') \in r(x)\}$.

Remark 9 *The correspondence between ultrapaths and filters is one-to-one, that is for ultrapaths y and z in $\mathfrak{p}$, $\mathcal{A}_y = \mathcal{A}_z$ if and only if $y = z$. Likewise for infinite paths γ and γ' in $\mathfrak{p}^\infty$, $\mathcal{A}_\gamma = \mathcal{A}_{\gamma'}$ if and only if $\gamma = \gamma'$. Furthermore, if y is an ultrapath and if γ is an infinite path, then $\mathcal{A}_y \neq \mathcal{A}_\gamma$.*

Proof. Recall that if y is an ultrapath the set $\mathcal{A}_y$ is given by $\mathcal{A}_y = \{(x, x) \in E(S_{\mathcal{G}}) : (x, x)(y, y) \neq \omega\ ;\ |x| < |y| \text{ or, if } |x| = |y|, \text{ then } r(x) \supseteq r(y)\}$. Obviously, if $y = z$ then $\mathcal{A}_y = \mathcal{A}_z$. So suppose that the equality, $\mathcal{A}_y = \mathcal{A}_z$, holds. Then $(y, y) \in \mathcal{A}_z$ and $(z, z) \in \mathcal{A}_y$. Hence the product in $E(S_{\mathcal{G}})$, $(y, y)(z, z)$, is not ω. Furthermore, since $(y, y) \in \mathcal{A}_z$, we see that $|y| \leq |z|$, and since $(z, z) \in \mathcal{A}_y$, it follows that $|y| \geq |z|$. Hence $|z| = |y|$. But again, since

$(y,y)\in\mathcal{A}_z$ it follows that $r(z)\subseteq r(y)$ and since $(z,z)\in\mathcal{A}_y$ it follows that then $r(y)\subseteq r(z)$. Consequently, the equality, $r(y)=r(z)$, also holds. To see that $y=z$, suppose first that one of the ultrapaths, y or z, is in $\mathcal{G}^0$. Then since $|y|=|z|$, it follows that the other is also in $\mathcal{G}^0$. But since $r(y)=r(z)$, it follows that $y=z$. Now suppose that y and z have positive length and recall that the inequality, $(y,y)(z,z)\neq\omega$, means that the paths y and z are comparable. (See Definition 5 and see the paragraph following equation (1).) Suppose that y has z as an initial segment. That is, suppose that $y=z\cdot y'$ for some $y'\in\mathfrak{p}$. (See the paragraph following equation (1).) Then since $|z|=|y|$, it follows that $y'\in\mathcal{G}^0$, which yields the equality, $y=z_{y'}$. (See equation (1).) But since, $r(y)=r(z)$, it follows that, $r(z)=r(y)=r(z_{y'})=r(z)\cap y'$. Thus, the inclusion, $r(z)\subseteq y'$, holds and hence, $y=z_{y'}=z$. (See the paragraph following equation (1).) A similar argument shows that $z=y$, in the case when, $z=y\cdot z'$ for some $z'\in\mathfrak{p}$.

Next recall that if γ is an infinite path, then $\mathcal{A}_\gamma=\{(x,x)\in E(S_\mathcal{G}):\gamma=x\cdot\gamma',\gamma'\in\mathfrak{p}^\infty,s(\gamma')\in r(x)\}$. So evidently if $\gamma=\gamma'$ then $\mathcal{A}_\gamma=\mathcal{A}_{\gamma'}$. Suppose, conversely, that $\mathcal{A}_\gamma=\mathcal{A}_{\gamma'}$ and write $\gamma=e_1e_2\ldots$ and $\gamma'=e_1'e_2'\ldots$. For $i\geq 1$, and write $\alpha_i:=e_1\ldots e_i$ and write $\alpha_i':=e_1'\ldots e_i'$. Then, of course, $|\alpha_i|=|\alpha_i'|=i$. Since $\mathcal{A}_\gamma=\mathcal{A}_{\gamma'}$, we see that $(\alpha_i,\alpha_i)\in\mathcal{A}_{\gamma'}$ and $(\alpha_i',\alpha_i')\in\mathcal{A}_\gamma$. Hence the equations, $\alpha_i'e_{i+1}'\ldots=\gamma'=\alpha_i\eta'$ and $\alpha_ie_{i+1}\ldots=\gamma=\alpha_i'\eta$, hold for some infinite paths η and η'. But since $|\alpha_i|=|\alpha_i'|$ it follows that $\alpha_i=\alpha_i'$. Since $i\geq 1$ was fixed but arbitrary, it follows that $e_i=e_i'$ for each i. Hence $\gamma=\gamma'$. The last assertion is clear, since if γ is an infinite path, then $\mathcal{A}_\gamma$ contains elements (x,x) with $|x|$ arbitrarily large. ■

The unit space $H_\mathcal{G}^{(0)}=\widehat{E(S_\mathcal{G})}$, of the universal groupoid $H=H_\mathcal{G}$ of $S=S_\mathcal{G}$ has an explicit parametrization given by the following proposition.

Proposition 10 *The set of semicharacters on $E(S_\mathcal{G})$, $H_\mathcal{G}^{(0)}=\widehat{E(S_\mathcal{G})}$, may be identified with the disjoint union $\mathfrak{p}\cup\mathfrak{p}^\infty\cup\{\omega\}$.*

Proof. Let $\chi\in\widehat{E(S_\mathcal{G})}$ and recall that $E(S_\mathcal{G})=\{(x,x):x\in\mathfrak{p}\}\cup\{\omega\}$. If $\chi(\omega)=1$, then since $(x,x)\,\omega=\omega$ for all $(x,x)\in E(S_\mathcal{G})$, we see that $\chi(x,x)=1$ for all (x,x). That is, χ is the constant non-zero semicharacter on $E(S_\mathcal{G})$. So the filter in $E(S_\mathcal{G})$ determined by χ, $\mathcal{A}_\chi$, is simply $\tilde{\omega}$. Thus $\chi=\omega$. So we may suppose that $\chi\neq\omega$. Then $\chi(\omega)=0$. Let $M:=\{|x|:(x,x)\in\mathcal{A}_\chi\}$. The strategy here is the following. We will parametrize the filter in $E(S_\mathcal{G})$, $\mathcal{A}_\chi=\{(x,x)\in E(S_\mathcal{G}):\chi(x,x)=1\}$, ([6, p.173-174]) by showing that $\mathcal{A}_\chi=\mathcal{A}_y$, for an ultrapath $y\in\mathfrak{p}$ if M is finite and by showing that $\mathcal{A}_\chi=\mathcal{A}_\gamma$ for a suitable $\gamma\in\mathfrak{p}^\infty$ if M is infinite. Then we identify χ either with y or γ. (See Remark 9.) To begin recall that the set, M, is defined by the equation, $M=\{|x|:(x,x)\in\mathcal{A}_\chi\}$.

Case I M is finite. In this case there an ultrapath y in $\mathfrak{p}$ so that $(y,y)\in\mathcal{A}_\chi$ and so that $|y|=\max M$. We show that $\mathcal{A}_y=\mathcal{A}_\chi$. For this end, take any $(x,x)\in\mathcal{A}_y$. Then by the definition of $\mathcal{A}_y$, the inequality, $(x,x)(y,y)\neq\omega$, holds, and either $|y|>|x|$, or if $|y|=|x|$, then the inclusion, $r(y)\subseteq r(x)$, holds. (See the paragraph following equations (2) and (3).) Hence

by Remark 8, the inequality, $(x,x) \geq (y,y)$, holds. This means that, $(x,x)(y,y) = (y,y)$, which implies the equation $\chi(x,x)\chi(y,y) = \chi(y,y)$. (See previous paragraph to Remark 8) Since (y,y) belongs to $\mathcal{A}_\chi$, we see that $\chi(x,x) = 1$. Hence, $(x,x) \in \mathcal{A}_\chi$, and $\mathcal{A}_y \subseteq \mathcal{A}_\chi$. On the other hand suppose that $(x,x) \in \mathcal{A}_\chi$. Since $\chi(\omega) = 0$, it follows that the product, in $\mathcal{A}_\chi$, $(x,x)(y,y)$, is not ω. (Recall that $(y,y) \in \mathcal{A}_\chi$) Moreover, the inequality, $|y| \geq |x|$, holds, since $|y| = \max M$. This yields the equation, $y = x \cdot y'$, for some $y' \in \mathfrak{p}$. If $|y| = |x|$, then $y' \in \mathcal{G}^0$ and hence we must have $y = x_{y'}$. But then $r(y) = r(x_{y'}) = r(x) \cap y' \subseteq r(x)$. (See equation (1) in Notation 4.) Thus $(x,x) \in \mathcal{A}_y$ and hence $\mathcal{A}_\chi \subseteq \mathcal{A}_y$. Thus $\mathcal{A}_\chi = \mathcal{A}_y$. (Note that Remark 9 shows that y is uniquely determined by χ.

Case II M is infinite. We'll show that there is a path $\gamma \in \mathfrak{p}^\infty$ such that $\mathcal{A}_\chi = \mathcal{A}_\gamma$. Indeed, take a pair $(x_1,x_1) \in \mathcal{A}_\chi$ such that $|x_1| > 0$. Since the set M is countably infinite, we may find another pair $(x_2,x_2) \in \mathcal{A}_\chi$, such that $|x_2| > |x_1|$. Moreover, since the product $(x_1,x_1)(x_2,x_2)$ is not ω, we may write $x_2 = x_1 \cdot y_2$, for some $y_2 \in \mathfrak{p}$ such that $|y_2| > 0$. Using the same argument for the pair (x_2,x_2), we may find another pair, $(x_3,x_3) \in \mathcal{A}_\chi$, such that $x_3 = x_2 \cdot y_3$ for some $y_3 \in \mathfrak{p}$ with $|y_3| > 0$. Continuing this process inductively, we obtain a sequence of pairs $\{(x_i,x_i)\}_{i\geq 1}$ in $\mathcal{A}_\chi$ so that each x_i has positive length and $x_{i+1} = x_i \cdot y_{i+1}$, where $y_i \in \mathfrak{p}$ with $|y_i| > 0$, for all i. So, if we set $x_i = (\alpha_i, A_i)$ and $y_i = (\beta_i, B_i)$, we may use the relation, $x_{i+1} = x_i \cdot y_{i+1}$, to define an infinite path γ in $\mathfrak{p}^\infty$ by the equation $\gamma := \alpha_1\beta_2\beta_3\ldots$. We show that $\mathcal{A}_\chi = \mathcal{A}_\gamma$. For this end, take any (x,x) in $\mathcal{A}_\chi$. Then the product in $\mathcal{A}_\chi$, $(x,x)(x_i,x_i)$, is not ω for any i. Since the set, $\{|x_i| : i = 1,\ldots\}$ is unbounded above, there is a positive integer i_0, such that $|x_{i_0}| > |x|$. Then by definition, we have $x_{i_0} = x \cdot x'_{i_0}$ for some $x'_{i_0} \in \mathfrak{p}$ with positive length. But by setting $x'_{i_0} = (\alpha'_{i_0}, A'_{i_0})$, we have $\gamma = x_{i_0} \cdot \beta_{i_0+1}\ldots = x \cdot \alpha'_{i_0}\beta_{i_0+1}\ldots$. (Note that $\gamma = x_i \cdot \beta_{i+1}\beta_{i+2}\ldots$, for each $i \geq 1$ by the paragraph before Definition 5.) Thus $(x,x) \in \mathcal{A}_\gamma$ and hence $\mathcal{A}_\chi \subseteq \mathcal{A}_\gamma$. For the reverse inclusion, let $(x,x) \in \mathcal{A}_\gamma$. Then $\gamma = x \cdot \gamma'$ for some infinite path γ' in $\mathfrak{p}^\infty$. So in this situation we always may choose a path x_{i_0} from the sequence $\{(x_i,x_i)\}_{i\geq 1}$, so that the product in $E(S_\mathcal{G})$, $(x,x)(x_{i_0},x_{i_0})$, is not ω and $|x_{i_0}| > |x|$. Then since $\gamma := \alpha_1\beta_2\beta_3\ldots = x_i \cdot \beta_{i+1}\ldots$ for each $i \geq 1$, Remark 8 shows that the inequality, $(x,x) > (x_{i_0},x_{i_0})$. This, in turn, yields the equation, $(x,x)(x_{i_0},x_{i_0}) = (x_{i_0},x_{i_0})$. It follows that $\chi(x,x)\chi(x_{i_0},x_{i_0}) = \chi(x_{i_0},x_{i_0})$, and since $\chi(x_{i_0},x_{i_0}) = 1$, it follows that $\chi(x,x) = 1$. Thus $(x,x) \in \mathcal{A}_\chi$, showing that $\mathcal{A}_\gamma \subseteq \mathcal{A}_\chi$. Hence $\mathcal{A}_\chi = \mathcal{A}_\gamma$. Again, we may appeal to Remark 9 to guarantee that the infinite path γ is uniquely determined by χ.

■

Recall that the topology on $\widehat{E(S_\mathcal{G})}$ is the topology of pointwise convergence on $E(S_\mathcal{G})$, and so the family, $\{D_{(x,x)} : (x,x) \in E(S_\mathcal{G})\}$, of compact open sets forms a subbasis for the topology. In this setting, the subbasic set, $D_{(x,x)}$, is

given by the equation, $D_{(x,x)} = \{y \in \mathfrak{p} : (x,x)(y,y) \neq \omega;\ |y| > |x|$ or, if $|y| = |x|$, then $r(y) \subseteq r(x)\} \cup \{\gamma \in \mathfrak{p}^{\infty} : \gamma = x \cdot \gamma',\ \gamma' \in \mathfrak{p}^{\infty}, s(\gamma') \in r(x)\}$. (See [6, Chap.4, p.174] and equations (2) and (3).) We would like a more concrete description of the topology on $\widehat{E(S_{\mathcal{G}})}$. For this purpose, it is convenient to introduce the following notation and definition.

Notation 11 *Let F be a finite subset of $E(S_{\mathcal{G}})$. Let $D_{(x,x);F} := D_{(x,x)} \cap \bigcap_{(z,z)\in F} D^c_{(z,z)}$. Since we are only interested in non-empty basic sets, we may suppose that $(z,z) < (x,x)$, for all $(z,z) \in F$. See [6, p.174].*

Definition 12 *Let $\mathcal{G} = \left(G^0, \mathcal{G}^1, r, s\right)$ be an ultragraph and let A be a subset of G^0. We say that the edge $e \in \mathcal{G}^1$ is emitted by A whenever $s(e) \in A$.*

Lemma 13 *Given an ultrapath $y \in D_{(x,x),F}$ there is a finite set K of edges emitted by the range of y, $r(y)$, and a finite subcollection Q of $\mathcal{G}^0$, such that no set in Q contains $r(y)$ and such that $y \in D_{(y,y);K,Q} \subset D_{(x,x),F}$ where,*

$$D_{(y,y);K,Q} := D_{(y,y)} \cap \bigcap_{e\in K} D^c_{(y\cdot e, y\cdot e)} \cap \bigcap_{C\in Q} D^c_{(y_C, y_C)}.$$

Proof. Recall that, the open subset of $\widehat{E(S_{\mathcal{G}})}$, $D_{(x,x)}$, is given by the equation, $D_{(x,x)} = \{y \in \mathfrak{p} : (x,x)(y,y) \neq \omega;\ |y| > |x|$ or, if $|y| = |x|$, then $r(y) \subseteq r(x)\} \cup \{\gamma \in \mathfrak{p}^{\infty} : \gamma = x \cdot \gamma',\ \gamma' \in \mathfrak{p}^{\infty}, s(\gamma') \in r(x)\}$. We leave to the reader to check that the ultrapath, y, lies in $D_{(y,y);K,Q}$. Suppose $y \in D_{(x,x),F}$ and fix any $(z,z) \in F$, such that $(z,z) < (x,x)$. Then since $y \in D_{(x,x),F}$, we have, $(x,x)(y,y) \neq \omega$, and so either $|y| > |x|$ or, if $|y| = |x|$, then $r(y) \subseteq r(x)$ and $y \notin D_{(z,z)}$. Also since $(z,z) < (x,x)$, we have $|z| > |x|$ or, if $|z| = |x|$, then $r(z) \subsetneq r(x)$. (See Remark 8.) We have the following cases:

Case I $|z| \geq 1$. Since $y \notin D_{(z,z)}$, if the equation, $(y,y)(z,z) = \omega$, holds, we have $D_{(y,y)} \subseteq D_{(x,x)} \cap D^c_{(z,z)}$; otherwise, either the inequality, $|y| < |z|$, holds or, if $|y| = |z|$, then the range of z, $r(z)$, does not contain the range of y, $r(y)$. In this situation, we have $z = y \cdot z'$, for some $z' \in \mathfrak{p}$. So we have to consider two subcases in this first case.

I(1) $|z'| \geq 1$. Let e' be the initial edge in $\mathcal{G}^1$ of z'. So $s(e') \in r(y)$, and we see that $D_{(y,y)} \cap D^c_{(y\cdot e', y\cdot e')} \subseteq D_{(x,x)} \cap D^c_{(z,z)}$.

I(2) $z' \in \mathcal{G}^0$. In this case, the equality, $|z| = |y|$, holds. So since $y \notin D_{(z,z)}$, the set $r(z) = r(y) \cap z'$, does not contain the range of y, $r(y)$. This implies that z' can not contain the range of y, $r(y)$. Then we see that $D_{(y,y)} \cap D^c_{(y_{z'}, y_{z'})} \subseteq D_{(x,x)} \cap D^c_{(z,z)}$.

Case II $z \in \mathcal{G}^0$. In this case, $x \in \mathcal{G}^0$ and $z \subsetneq x$. But since $y \notin D_{(z,z)}$, $s(y) \nsubseteq z$. Then we have, $D_{(y,y)} \subseteq D_{(x,x)} \cap D^c_{(z,z)}$.

In any of the above cases, we may take the set K to be the union when (z,z) runs over the set F of the sets $\{e'_z \in \mathcal{G}^1\colon\ e'_z$ is the initial edge of z', $z = y \cdot z'\}$; while for the set Q, we may take the set, $\{z' \in \mathcal{G}^0\colon\ z = y_{z'},\ (z,z) \in F\}$. Thus $y \in D_{(y,y);K,Q} \subset D_{(x,x),F}$. ■

The following lemma describes the topology on the set of infinite paths $\mathfrak{p}^{\infty}$.

Lemma 14 *A neighborhood basis for $\gamma \in \mathfrak{p}^{\infty}$ is given by the sets of the form $D_{(y,y)}$, where $y = (\beta, B)$ and β is an initial segment of γ.*

Proof. Let $\gamma \in \mathfrak{p}^{\infty}$ and $\gamma \in D_{(x,x);F} := D_{(x,x)} \cap \bigcap_{(z,z)\in F} D^c_{(z,z)}$. Since $\gamma \in D_{(x,x)}$, $\gamma = x \cdot \gamma'$, where $\gamma' \in \mathfrak{p}^{\infty}$ and $s(\gamma') \in r(x)$. Also since we are interested in nonempty basis elements, we may assume that the inequality, $(x,x) > (z,z)$, holds. So we have $z = x \cdot z'$, $z' \in \mathfrak{p}$. Since γ' is an infinite path and $\gamma \notin D_{(z,z)}$, we may choose an initial segment with positive length, y', of γ' so that $|y'| > |z'|$. Set $y := x \cdot y'$ and notice then $\gamma \in D_{(y,y)} \subset D_{(x,x)} \cap D^c_{(z,z)}$. ■

The universal groupoid $H_{\mathcal{G}}$ for $S_{\mathcal{G}}$ has an explicit parametrization given by the following theorem.

Theorem 15 *The universal groupoid $H_{\mathcal{G}}$ for $S_{\mathcal{G}}$ can be identified with the union of $\{\omega\}$ and the set of all triples of the form $(x \cdot \mu, |x| - |y|, y \cdot \mu)$ where $x,\ y \in \mathfrak{p}$, $r(x) = r(y)$, $\mu \in \mathfrak{p} \cup \mathfrak{p}^{\infty}$, and $x \cdot \mu,\ y \cdot \mu \in \mathfrak{p} \cup \mathfrak{p}^{\infty}$. Multiplication on $H_{\mathcal{G}}$ is given by the formula:*

$$(x \cdot \mu, |x| - |y|, y \cdot \mu)(y \cdot \mu, |y| - |y'|, y' \cdot \mu') := (x \cdot \mu, |x| - |y'|, y' \cdot \mu'),$$

and inversion is given by the formula,

$$(x \cdot \mu, |x| - |y|, y \cdot \mu)^{-1} := (y \cdot \mu, |y| - |x|, x \cdot \mu).$$

The canonical map $\Lambda : S_{\mathcal{G}} \longrightarrow H^a$, sends ω to $\{\omega\}$ and any element $(x,y) \in S_{\mathcal{G}}$ to the (compact open) set $\mathcal{A}(x,y)$, where

$$\mathcal{A}(x,y) := \{(x \cdot \mu, |x| - |y|, y \cdot \mu) : \mu \in \mathfrak{p} \cup \mathfrak{p}^{\infty}\}.$$

Further $\mathcal{A}(x,x) = D_{(x,x)}$, for each $x \in \mathfrak{p}$. The locally compact groupoid $H_{\mathcal{G}}$ is Hausdorff.

Proof. The proof is close to that for the Cuntz semigroup S_n in [6, p.182-186]. Also see [7, p.9-10]. We have to compute the equivalence classes $\overline{(\chi, s)}$, $\chi \in D_s$. We suppose first $\chi = u$, is an ultrapath and then the computation of the equivalence class $\overline{(\chi, s)}$, when χ is an infinite path is similar. So let $\chi = u$, and let $s = (x,y)$ and $t = (z,w)$ be elements in $S_{\mathcal{G}}$ such that $\chi \in D_{(x,x)(z,z)}$. Then $u = x \cdot u'$ and $u = z \cdot u''$. If we let $u = (u_1 \cdots u_{|u|}, U_{|u|})$, then without lost of generality, we may assume that $x = u_1 \cdots u_m$, and $z = u_1 \cdots u_r$, where $m \leq r \leq |u|$. Thus $s = (u_1 \cdots u_m, y)$ and $t = (u_1 \cdots u_r, w)$. Let $e \in E(S_{\mathcal{G}})$ be such that $\chi = u \in D_e$, $e \leq (ss^*)(tt^*) = (x,x)(z,z) = (z,z)$ and $es = et$. Then $e = (f,f)$, where f is such that $u = f \cdot u'''$, $u''' \in \mathfrak{p}$ and $f = z \cdot f''$, $f'' \in \mathfrak{p}$. So

we can write $e = (f, f) = (u_1 \cdots u_{r'}, u_1 \cdots u_{r'})$, where $m \le r \le r' \le |u|$. Then we have

$$\begin{aligned}
&((u_1 \cdots u_{r'}, y \cdot u_{m+1} \cdots u_{r'}) \\
&= (f, y \cdot f') = es = et \\
&= (f, w \cdot f'') = ((u_1 \cdots u_{r'}, w \cdot u_{r+1} \cdots u_{r'}) .
\end{aligned}$$

This means that $w = y \cdot u_1 \cdots u_{r'}$. To link our groupoid with Renault's model for the Cuntz groupoid G_n, described in [6, Section 4.2, Example 3], we associate the pair

$$(\chi, s) = ((u_1 \cdots u_m u_{m+1} \cdots u_{|u|}, U_{|u|}), (u_1 \cdots u_m, y)),$$

with the triple

$$((u_1 \cdots u_m u_{m+1} \cdots u_{|u|}, U_{|u|}), m - |y|, y \cdot (u_{m+1} \cdots u_{|u|}, U_{|u|})).$$

The argument is reversible and shows that this map is a bijection.

We now prove that $H_{\mathcal{G}}$ is Hausdorff, leaving the remaining verifications of the theorem to the reader. Let $a = (x, |x| - |y|, y)$, $b = (x', |x'| - |y'|, y')$ belong to $H_{\mathcal{G}}$ with $a \ne b$. If $\mathcal{A}(x, y) \cap \mathcal{A}(x', y') = \emptyset$, then we can separate a and b using $\mathcal{A}(x, y)$ and $\mathcal{A}(x', y')$. Suppose that $\mathcal{A}(x, y) \cap \mathcal{A}(x', y') \ne \emptyset$. Then there exist χ, χ' such that

$$(x \cdot \chi, |x| - |y|, y \cdot \chi) = (x' \cdot \chi', |x'| - |y'|, y' \cdot \chi') .$$

Then the equation, $|x| - |y| = |x'| - |y'|$, holds. Furthermore the ultrapaths x, x' and y, y' are comparable. We can suppose that for some $u \in \mathfrak{p}$, $x' = x \cdot u$. Then $y' = y \cdot w$, $w \in \mathfrak{p}$, where $|u| = |w|$, and since u and w are initial segments of χ, $u = w$. Then $x' = x \cdot u$ and $y' = y \cdot u$. If $|u| > 0$ then $\mathcal{A}(x, y) \cap \mathcal{A}^c(x', y')$ and $\mathcal{A}(x', y')$ separate a and b. If $|u| = 0$ then, $|x'| = |x|$ and $|y'| = |y|$. But since $a \ne b$, we have $r(x') \subsetneq r(x)$ and $r(y') \subsetneq r(y)$. So we see that $\mathcal{A}(x, y) \cap \mathcal{A}^c(x', y')$ and $\mathcal{A}(x', y')$ separate a and b as well. So $H_{\mathcal{G}}$ is Hausdorff. ■

Remark 16 *The singleton $\{\omega\}$ is a clopen invariant subset of $H_{\mathcal{G}}^{(0)}$. The reduction $H_{\mathcal{G}}^{(0)}|_{\{\omega\}}$ is simply $\{\omega\}$. Consequently, the interesting part of $H_{\mathcal{G}}$ is $H_{\mathcal{G}}|_{\mathfrak{p} \cup \mathfrak{p}^\infty}$.*

At this point, we have just identified the universal groupoid $H_{\mathcal{G}}$ for $S_{\mathcal{G}}$. However, we still need to find the correct groupoid for $\mathcal{G}$. That is, we want to find a groupoid $\mathfrak{G}_{\mathcal{G}}$ such that $C^*(\mathcal{G}) \backsimeq C^*(\mathfrak{G}_{\mathcal{G}})$. (See Definition 2 and Remark 3.) To do this we shall take a closer look at the unit space of $H_{\mathcal{G}}$, which is the disjoint union $\mathfrak{p} \cup \mathfrak{p}^\infty \cup \{\omega\}$, and use the concept of *ultrafilters* to investigate it. (See [9].) Indeed, we give G^0 the discrete topology. Then the points of the Stone-Čech compactification βG^0 of G^0 can be regarded as ultrafilters on G^0. (See the introduction in [3].) So, consider the subcollection of $\mathcal{G}^0$, $\mathcal{U}(\mathcal{G}^0)$, defined to be the collection of all sets in $\mathcal{G}^0$ whose principal filter in $\mathcal{G}^0$ is also

an ultrafilter over G^0. That is, $\mathcal{U}\left(\mathcal{G}^0\right) = \{A \in \mathcal{G}^0 : \widetilde{A} \in \beta G^0\}$, where $\widetilde{A}$ is the principal filter in $\mathcal{G}^0$ determined by $A \in \mathcal{G}^0$, i.e., $\widetilde{A} = \{B \in \mathcal{G}^0 \mid A \subseteq B\}$. (See the paragraph following Remark 8.) Observe that $\mathcal{U}\left(\mathcal{G}^0\right)$ contains every singleton set determined by the vertices in G^0. Furthermore, one can check that a topology for $\mathcal{U}\left(\mathcal{G}^0\right)$ may defined by taking the family of subsets of $\mathcal{U}\left(\mathcal{G}^0\right)$, $\{\widehat{A} : A \in \mathcal{G}^0\}$, where $\widehat{A} := \{B \in \mathcal{U}\left(\mathcal{G}^0\right) : A \in \widetilde{B}\}$, as a subbasis of closed subsets. (See [9, first paragraph on page 117].) An important property of the collection, $\mathcal{U}\left(\mathcal{G}^0\right)$, however, is that, for each member C in $\mathcal{U}\left(\mathcal{G}^0\right)$, its associated semicharacter, χ_C, satisfies the equations:

$$\chi_C\left(A \cup B\right) = \chi_C\left(A\right) + \chi_C\left(B\right) - \chi_C\left(A \cap B\right) \text{ and } \chi_C\left(\emptyset\right) = 0, \tag{4}$$

for all $A, B \in \mathcal{G}^0$. (See the paragraph following Remark 8 and see [9, p.104].) Thus, $\mathcal{U}\left(\mathcal{G}^0\right)$, may be viewed as a closed subset of $\widetilde{\mathcal{G}^0}$ via the map $A \longrightarrow \chi_A$. As we shall see, this observation is critical for building the groupoid model for $\mathcal{G}$. (See second part of *(i)* in Definition 2 and Remark 3). We shall call the elements in $\mathcal{U}\left(\mathcal{G}^0\right)$ *ultrasets*. Another crucial tool is the next proposition, which identifies the closure of the set of all infinite paths $\mathfrak{p}^\infty$, $\overline{\mathfrak{p}^\infty}$. This is the key to obtaining the unit space of our groupoid for $\mathcal{G}$. (Compare with the first statement of Proposition 4 in [7].) For this purpose, we need the following generalization of the notion of "infinite emitter" from the setting of ordinary graphs.

Definition 17 *Let $\mathcal{G} = \left(G^0, \mathcal{G}^1, r, s\right)$ be an ultragraph and for each subset A of G^0, let $\varepsilon\left(A\right)$ be the set $\{e \in \mathcal{G}^1 : s\left(e\right) \in A\}$. We shall say that a set A in $\mathcal{G}^0$ is an* infinite emitter *whenever $\varepsilon\left(A\right)$ is infinite.*

Proposition 18 *The set of infinite paths, $\mathfrak{p}^\infty$, is dense in $Y_\infty \cup \mathfrak{p}^\infty$, where Y_∞ is defined to be the set of all ultrapaths y in $\mathfrak{p}$ whose range $r\left(y\right)$ is an ultraset emitting infinitely many edges.*

Proof. Take any χ in the closure $\overline{\mathfrak{p}^\infty}$ in $\widehat{E\left(S_{\mathcal{G}}\right)} = \mathfrak{p} \cup \mathfrak{p}^\infty \cup \{\omega\}$. Then there is an infinite sequence $\{\gamma_i\}_{i \geq 1}$ in $\mathfrak{p}^\infty$ such that $\gamma_i \longrightarrow \chi$. If χ is not an infinite path, then by Remark 16 it must be an ultrapath, say $\chi = y$. So for large i, each $\gamma_i = y \cdot \gamma_i'$ where, $\{s(\gamma_i')\} \longrightarrow r\left(y\right)$ eventually. (See Lemma 14.) Since $\mathcal{U}\left(\mathcal{G}^0\right)$ is closed and since each $\{s(\gamma_i')\}$ lies in $\mathcal{U}\left(\mathcal{G}^0\right)$, it follows that $r\left(y\right)$ belongs to $\mathcal{U}\left(\mathcal{G}^0\right)$. Therefore $r\left(y\right) \in \widehat{r\left(y\right)}$. Moreover, since $\{s(\gamma_i')\} \longrightarrow r\left(y\right)$ eventually, it follows that $\{s(\gamma_i')\} \in \widehat{r\left(y\right)}$, for infinitely many i's. Thus $r\left(y\right)$ is an infinite emitter. For the reverse inclusion, take any y in Y_∞. Then by the definition of Y_∞, the range of y, $r\left(y\right)$, is an ultraset emitting infinitely many edges. Suppose that the ultrapath y belongs to the open set in $\widehat{E\left(S_{\mathcal{G}}\right)}$, $D_{(y,y);K,Q}$, which is,

$$D_{(y,y)} \cap \bigcap_{e \in K} D^c_{(y \cdot e, y \cdot e)} \cap \bigcap_{C \in Q} D^c_{(y_C, y_C)},$$

where the set K is a finite set of edges emitted by $r\left(y\right)$, and Q is a finite subcollection of $\mathcal{G}^0$ consisting of sets that do not contain the range of y, $r\left(y\right)$.

(See [6, Chap.4, p.174] and Lemma 13.) Then no set C in Q belongs to $\widetilde{r(y)}$. (Recall that, $\widetilde{r(y)} = \left\{A \in \mathcal{G}^0 : r(y) \subseteq A\right\}$.) Fix a set C in Q. Then since $C \notin \widetilde{r(y)}$ and since $\widetilde{r(y)}$ is an ultrafilter on G^0, the complement of C, C^c, must belong to $\widetilde{r(y)}$. (See [9, Theorem IV, p.107].) That is, $r(y)$ and C are disjoint. But by hypothesis the set, $r(y)$, is an infinite emitter. Consequently, the set, $\varepsilon(r(y))$, is infinite. So we always may choose an edge e_1 in $\mathcal{G}^1$, such that $e_1 \notin K$ and $s(e_1) \in r(y) \subseteq C^c$. Since we are assuming that the ultragraph $\mathcal{G}$ has no sinks, we may choose another edge, say e_2, so that $s(e_2) \in r(e_1)$. Inductively, we may form an infinite path, $\gamma := e_1 e_2 \ldots$, so that $s(\gamma) \in r(y) \subseteq C^c$. Since C was fixed but arbitrary, we may conclude that the source of γ, $s(\gamma)$, belongs to $r(y)$ but not to any set C in Q. So setting $\gamma' = y \cdot \gamma$, we see that $\gamma' \in D_{(y,y);K,Q}$, and hence, we may conclude that y lies in the closure of $\mathfrak{p}^\infty$, $\overline{\mathfrak{p}^\infty}$. ■

We next set $X := Y_\infty \cup \mathfrak{p}^\infty \subset H_{\mathcal{G}}^{(0)}$. By Proposition 18, X is a closed subset of the unit space $H_{\mathcal{G}}^{(0)} = \mathfrak{p} \cup \mathfrak{p}^\infty \cup \{\omega\}$. Hence X is a locally compact Hausdorff space. Furthermore, since for every triple, $(x \cdot \mu, |x| - |y|, y \cdot \mu)$, in $H_{\mathcal{G}}$, the equation, $r(x) = r(y)$, holds, it follows that X is also an invariant subset of $H_{\mathcal{G}}^{(0)}$. Let $\mathfrak{G}_{\mathcal{G}}$ be the reduction of $H_{\mathcal{G}}$ to X, i.e. let $\mathfrak{G}_{\mathcal{G}} = H_{\mathcal{G}}|_X$. (Compare with the first paragraph following the proof of Theorem 1 in [7].) Then $\mathfrak{G}_{\mathcal{G}}$ is a closed subgroupoid of $H_{\mathcal{G}}$, and is an r-discrete groupoid with counting measures giving a left Haar system. We will call $\mathfrak{G}_{\mathcal{G}}$ the *ultrapath groupoid of* $\mathcal{G}$.

For $(x, y) \in S_{\mathcal{G}}$, we define, $\mathcal{A}'(x, y) = \mathcal{A}(x, y) \cap \mathfrak{G}_{\mathcal{G}}$, $\mathcal{A}'(x, x) = D_{(x,x)} \cap X$, and $\mathcal{A}'(\omega) = \{\omega\} \cap X = \emptyset$. Then each $\mathcal{A}'(s)$, $s \in S_{\mathcal{G}}$, is a compact as well as open subset of $\mathfrak{G}_{\mathcal{G}}$.

Recall that $C^*(\mathfrak{G}_{\mathcal{G}})$ is the completion of the space $C_c(\mathfrak{G}_{\mathcal{G}})$ of all continuous complex-valued functions on $\mathfrak{G}_{\mathcal{G}}$ with compact support, with respect to the norm $||f|| = \sup_\pi ||\pi(f)||$, where the supremum is taken over all I-norm continuous representations π of $C_c(\mathfrak{G}_{\mathcal{G}})$, (see [6, p.101] and [8, Definition1.5].) For each $A \in \mathcal{G}^0$, let $q_A := 1_{\mathcal{A}'(A,A)}$ and for $e \in \mathcal{G}^1$, let $t_e := 1_{\mathcal{A}'((e,r(e)),r(e))}$. We will show that the family $\left\{t_e, q_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\right\}$, of characteristic functions on $\mathfrak{G}_{\mathcal{G}}$ is a Cuntz-Krieger $\mathcal{G}$-family in the groupoid C^*-algebra, $C^*(\mathfrak{G}_{\mathcal{G}})$. Before we prove our assertion, we clarify our calculations via the following lemma whose proof we leave to the reader.

Lemma 19 *Let A, $B \in \mathcal{G}^0$. Observe that $\mathcal{A}'(A, A) = \{\mu \in X : s(\mu) \subseteq A\} = \{\mu \in X : A \in \widetilde{s(\mu)}\}$ and, $\mathcal{A}'(e, e) = \{e \cdot \mu \in X : \mu \in \mathfrak{p} \cup \mathfrak{p}^\infty\}$. (See Notation 4) Then:*

(1) $\mathcal{A}'(A \cap B, A \cap B) = \mathcal{A}'(A, A) \cap \mathcal{A}'(B, B)$;

(2) $\mathcal{A}'(A \cup B, A \cup B) = \mathcal{A}'(A, A) \cup \mathcal{A}'(B, B)$;

(3) $\mathcal{A}'(A, A) = \bigcup_{s(e) \in A} \mathcal{A}'(e, e) \cup G'(A)$*, where* $G'(A) := \{B \in X : B \subseteq A\}$.

To check assertion *(2)* of Lemma 19, the reader may use Theorem V, Page 117 in [9] and the fact that a set $B \in \mathcal{G}^0$ that lies in X is such that its principal filter, $\widetilde{B}$, is an ultrafilter over G^0.

Proposition 20 *The family* $\{t_e, q_A : A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$ *is a Cuntz-Krieger* $\mathcal{G}$*-family in* $C^*(\mathfrak{G}_{\mathcal{G}})$.

Proof. We leave to the reader to check that the t_e's are partial isometries with mutually orthogonal ranges and the q_A's are projections. Also it is easy to verify that $t_e^* t_e = q_{r(e)}$ and $t_e t_e^* \leq q_{s(e)}$. Then we have $q_\emptyset = 1_{\mathcal{A}'(\emptyset,\emptyset)} = 1_\emptyset = 0$. Also for all A, $B \in \mathcal{G}^0$, we see by (1) and (2) in Lemma 19 that the equations, $q_{A\cap B} = q_A q_B$ and $q_{A\cup B} = q_A + q_B - q_{A\cap B}$ hold. (Recall Remark 3.) Finally let $v \in G^0$ such that $0 < |s^{-1}(v)| < \infty$. Then by (3) of Lemma 19, $q_v - \sum_{s(e)=v} t_e t_e^* = 1_{\mathcal{A}'(v,v)} - \sum_{s(e)=v} 1_{\mathcal{A}'(e,e)} = 1_{G'(\{v\})}$. But since $0 < |s^{-1}(v)| < \infty$, it follows that, $\{v\} \notin X$. Consequently, $G'(\{v\}) = \emptyset$, and hence $q_v - \sum_{s(e)=v} t_e t_e^* = 0$. ■

The following straightforward lemma tells us that the collection of compact open subsets of $\mathfrak{G}_{\mathcal{G}}$, $\{\mathcal{A}'(x,y) : x, y \in S_{\mathcal{G}}\}$, is a subbasis for the topology of $\mathfrak{G}_{\mathcal{G}}$.

Lemma 21 *Given* (x,y) *and* (z,w) *in* $S_{\mathcal{G}}$*, then*

$$\mathcal{A}'(x,y) \cap \mathcal{A}'(z,w) = \begin{cases} \mathcal{A}'(x,y) & \text{if } x = z\cdot x',\ y = w\cdot x',\ \text{for some } x' \in \mathfrak{p}; \\ \mathcal{A}'(z,w) & \text{if } z = x\cdot z',\ w = y\cdot z',\ \text{for some } z' \in \mathfrak{p}; \\ \mathcal{A}'(z,z) & \text{if } x = y \in \mathcal{G}^0, z = w,\ s(z) \in x, |z| \geq 1; \\ \emptyset & \text{otherwise} \end{cases}$$

Next define $\Lambda' : S_{\mathcal{G}} \longrightarrow \mathfrak{G}^a_{\mathcal{G}}$, by setting $\Lambda'(x,y) = \mathcal{A}'(x,y)$. Since X is closed and invariant subset of $H^{(0)}_{\mathcal{G}}$, Λ' is a well defined inverse semigroup isomorphism from $S_{\mathcal{G}}$ into the ample inverse semigroup $\mathfrak{G}^a_{\mathcal{G}}$. Then $\Lambda'(S_{\mathcal{G}})$ is an inverse subsemigroup of $\mathfrak{G}^a_{\mathcal{G}}$ which is a subbasis for the topology of $\mathfrak{G}_{\mathcal{G}}$ by Lemma 21. In fact the span W of characteristic functions $1_{\mathcal{A}'(x,y)}$ for $(x,y) \in S_{\mathcal{G}}$, is I-norm dense in $C_c(\mathfrak{G}_{\mathcal{G}})$, [6, Proposition 2.2.7]. Let $\{s_e, p_A : e \in \mathcal{G}^1, A \in \mathcal{G}^0\}$ be a universal Cuntz-Krieger $\mathcal{G}$-family in $C^*(\mathcal{G})$ and define a map ψ on the set of generators of $C^*(\mathcal{G})$ into $C^*(\mathfrak{G}_{\mathcal{G}})$, by the equations:

1. $\psi\left(s_\alpha p_A s_\beta^*\right) := 1_{\mathcal{A}'(x,y)}$, where $x = (\alpha, r(\alpha) \cap r(\beta) \cap A)$ and $y = (\beta, r(\alpha) \cap r(\beta) \cap A)$;
2. $\psi(s_\alpha p_A) := 1_{\mathcal{A}'(x,r(x))}$, where $x = (\alpha, r(\alpha) \cap A)$; and
3. $\psi(p_A) := 1_{\mathcal{A}'(A,A)}$, $A \in \mathcal{G}^0$.

Observe that $s_\alpha p_A s_\beta^* \neq 0$ precisely when $r(\alpha) \cap r(\beta) \cap A \neq \emptyset$. Then ψ extends to a surjective homomorphism which we shall denote also by ψ, such that $\psi(s_e) = 1_{\mathcal{A}'((e,r(e)),r(e))}$ and $\psi(p_A) = 1_{\mathcal{A}'(A,A)}$. Moreover, since we are assuming that $\mathcal{G}$ has no sinks, the inequality, $\psi(p_A) \neq 0$, holds for all nonempty sets A in $\mathcal{G}^0$. Let γ be the gauge action for $C^*(\mathcal{G})$, see [10, p.7]. For $z \in \mathbb{T}$ define $\tau_z : C^*(\mathfrak{G}_{\mathcal{G}}) \longrightarrow C^*(\mathfrak{G}_{\mathcal{G}})$, by the equation, $\tau_z(f)(h) = z^{c(h)} f(h)$, where $f \in C_c(\mathfrak{G}_{\mathcal{G}})$, $h \in \mathfrak{G}_{\mathcal{G}}$ and $c : \mathfrak{G}_{\mathcal{G}} \longrightarrow \mathbb{Z}$, is the cocycle defined by $c((x,k,x')) = k$, $k \in \mathbb{Z}$. Notice that τ is a strongly continuous action of $\mathbb{T}$ on $C^*(\mathfrak{G}_{\mathcal{G}})$. Also

by Proposition 20 the collection, $\{1_{\mathcal{A}'(A,A)}, 1_{\mathcal{A}'((e,r(e)),r(e))} : A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$, is a Cuntz-Krieger $\mathcal{G}$- family in $C^*(\mathfrak{G}_{\mathcal{G}})$. A simple computation shows that, $\psi \circ \gamma_z(s_e) = \tau_z \circ \psi(s_e)$ for all $e \in \mathcal{G}^1$, and $\psi \circ \gamma_z(p_A) = \tau_z \circ \psi(p_A)$, for all $A \in \mathcal{G}^0$. Thus the equation $\psi \circ \gamma_z = \tau_z \circ \psi$, holds for all $z \in \mathbb{T}$. By the Gauge-Invariant Uniqueness Theorem for ultragraphs, [10, Theorem 6.8], ψ is faithful and hence an isomorphism from $C^*(\mathcal{G})$ onto $C^*(\mathfrak{G}_{\mathcal{G}})$.

We may thus summarize our analysis to this point in the following theorem, which is a corollary to Theorem 7.

Theorem 22 *If $\mathcal{G}$ is an ultragraph without sinks, then $C^*(\mathcal{G}) \simeq C_0^*(S_{\mathcal{G}}) \simeq C^*(\mathfrak{G}_{\mathcal{G}})$.*

3 ULTRAGRAPH GROUPOIDS ARE AMENABLE

Let G be a discrete group, let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph and let $\varphi : \mathcal{G}^1 \longrightarrow G$ be a function. We introduce an analog of the skew product graph considered in [4]; the resulting object, which we denote by $\mathcal{G} \times_\varphi G$, is also an ultragraph. We show that the crossed product of $C^*(\mathcal{G})$ by the gauge action, $C^*(\mathcal{G}) \rtimes_\gamma \mathbb{T}$, is isomorphic to $C^*(\mathcal{G} \times_\varphi \mathbb{Z})$, where φ is the constant function 1 on $\mathcal{G}^1$. In this case, we shall write $\mathcal{G} \times_\varphi \mathbb{Z}$ as $\mathcal{G} \times_1 \mathbb{Z}$. It turns out that the ultragraph $\mathcal{G} \times_1 \mathbb{Z}$ has no loops and so by Theorem 4.1 in [11], $C^*(\mathcal{G} \times_1 \mathbb{Z})$ is an AF-algebra. It will then follow that $C^*(\mathcal{G}) \rtimes_\gamma \mathbb{T}$ is AF and, consequently, that $\mathfrak{G}_{\mathcal{G}}$ is amenable.

Definition 23 *Let G be a discrete group and let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ an ultragraph. Given a function $\varphi : \mathcal{G}^1 \longrightarrow G$ then the **skew product ultragraph** $\mathcal{G} \times_\varphi G$ is defined as follows: the set of vertices is $G^0 \times G$, the set of edges is $\mathcal{G}^1 \times G$ and the structure maps $s' : \mathcal{G}^1 \times G \longrightarrow \mathcal{G}^0 \times G$ and $r' : \mathcal{G}^1 \times G \longrightarrow P(\mathcal{G}^0 \times G)$, are defined by the equations,*

$$s'(e,g) = (s(e), g) \text{ and } r'(e,g) = r(e) \times \{g\varphi(e)\}.$$

We write $\mathcal{G} \times_\varphi G$ for $(G^0 \times G, \mathcal{G}^1 \times G, r', s')$.

It is clear that $\mathcal{G} \times_\varphi G$ is an ultragraph.

Remark 24 *We note that the ultragraph $\mathcal{G}$ has no singular vertices if and only if the skew ultragraph $\mathcal{G} \times_\varphi G$ has no singular vertices.*

Proof. This follows from the fact that, for any $v \in G^0$ and any $g \in G$, $(s')^{-1}(v,g) = s^{-1}(v) \times \{g\}$. ∎

We may assume that $\mathcal{G}$ has no singular vertices, i.e. no vertices which are infinite emitters. The reason for this is that given an ultragraph $\mathcal{G}$, one may build a new ultragraph $\mathcal{F}$ that has no singular vertices [10, Prop.6.2, p.17] such that $C^*(\mathcal{G})$ is strongly Morita equivalent to $C^*(\mathcal{F})$. ($\mathcal{F}$ is called the *desingularization* of $\mathcal{G}$.) Further, since $C^*(\mathcal{F})$ is AF, as we shall see, and since the property of being AF is preserved under strong Morita equivalence, we may conclude that $C^*(\mathcal{G})$ is AF. See also [10, Proposition 6.6.].

Proposition 25 *If $\mathcal{G}$ is an ultragraph with no singular vertices, then the unit space $\mathfrak{G}_{\mathcal{G}}^{(0)}$ of its groupoid model $\mathfrak{G}_{\mathcal{G}}$ becomes $\mathfrak{p}^{\infty}$, where $\mathfrak{p}^{\infty}$ denotes the infinite path space of $\mathcal{G}$.*

Proof. The unit space, $\mathfrak{G}_{\mathcal{G}}^{(0)}$, of $\mathfrak{G}_{\mathcal{G}}$ is the set $Y_{\infty} \cup \mathfrak{p}^{\infty} = \overline{\mathfrak{p}^{\infty}}$. Take any y in Y_{∞}. If $0 < |y|$, then the set, $\mathcal{A}'(\{s(y)\}, \{s(y)\})$, is an open subset of $\mathfrak{G}_{\mathcal{G}}^{(0)}$ which contains y. Thus there is an infinite sequence of infinite paths γ_k in $\mathfrak{p}^{\infty}$ such that $s(\gamma_k) = s(y)$ for large k. Thus, the vertex, $s(y)$, is an infinite emitter and hence a singular vertex in $\mathcal{G}$, contrary to hypothesis. If $|y| = 0$, say $y = A$ in $\mathcal{G}^0$, then set $\varepsilon(A)$ is infinite and $\widetilde{A}$ is an ultrafilter over G^0. (See Proposition 18) Therefore the set A must be infinite. (Otherwise $\mathcal{G}$ would have a singular vertex as well.) By Lemma 2.12 in [10], there are finite subsets $Y_1, \ldots, Y_n$ of $\mathcal{G}^1$ and a finite subset F of G^0 such that $A = \bigcap_{e \in Y_1} r(e) \cup \ldots \cup \bigcap_{e \in Y_n} r(e) \cup F$. Furthermore, F may be chosen to be disjoint from $\bigcap_{e \in Y_1} r(e) \cup \ldots \cup \bigcap_{e \in Y_n} r(e)$. So there is a finite number of edges $e_1, \ldots, e_n$ in the sets $Y_1, \ldots, Y_n$, respectively, such that $A \subseteq r(e_1) \cup \ldots \cup r(e_n)$. Therefore the set, $r(e_1) \cup \ldots \cup r(e_n)$, lies in the ultrafilter, $\widetilde{A}$. Hence by Theorem V in [9], we have, $r(e_i) \in \widetilde{A}$ for some $i \in \{1, \ldots, n\}$. Thus the inclusion, $A \subseteq r(e_i)$, holds. But then, the ultrapath, (e_i, A), would be in Y_{∞}. Then, as in the case when the length of y, $|y|$, is positive, the vertex, $s(e_i)$, will contradict the condition that $\mathcal{G}$ has no singular vertices. Therefore $Y_{\infty} = \emptyset$ and hence $\mathfrak{G}_{\mathcal{G}}^{(0)} = \mathfrak{p}^{\infty}$. ■

Recall that the position cocycle $c : \mathfrak{G}_{\mathcal{G}} \longrightarrow \mathbb{Z}$ is given by the formula $c(\chi, k, \chi') = k$, $k \in \mathbb{Z}$. In the following theorem we show that the skew product groupoid obtained from c, $\mathfrak{G}_{\mathcal{G}} \times_c \mathbb{Z}$, (see [8]) is the same as the path groupoid $\mathfrak{G}_{\mathcal{G} \times_1 \mathbb{Z}}$ of the skew ultragraph $\mathcal{G} \times_1 \mathbb{Z}$.

Theorem 26 *Let $\mathcal{G} = \left(G^0, \mathcal{G}^1, r, s\right)$ be an ultragraph with no singular vertices. Let $G = (\mathbb{Z}, +)$ be the discrete group of the integers under addition and let $\varphi : \mathcal{G}^1 \longrightarrow \mathbb{Z}$ be the function defined by $\varphi(e) = 1$ for $e \in \mathcal{G}^1$. Then the groupoid model $\mathfrak{G}_{\mathcal{G} \times_1 \mathbb{Z}}$ for $\mathcal{G} \times_1 \mathbb{Z}$ is isomorphic to the skew product groupoid $\mathfrak{G}_{\mathcal{G}} \times_c \mathbb{Z}$, where $c : \mathfrak{G}_{\mathcal{G}} \longrightarrow \mathbb{Z}$ is the position cocycle on $\mathfrak{G}_{\mathcal{G}}$.*

Proof. Since $\mathcal{G}$ has no singular vertices, we may identify the unit space of $\mathfrak{G}_{\mathcal{G}}$ with $\mathfrak{p}^{\infty}$ by Proposition 18. Next, we identify the unit space $\mathfrak{p}_{\mathcal{G} \times_1 \mathbb{Z}}^{\infty}$ of $\mathfrak{G}_{\mathcal{G} \times_1 \mathbb{Z}}$ with the unit space $\mathfrak{p}^{\infty} \times \mathbb{Z}$ of $\mathfrak{G}_{\mathcal{G}} \times_c \mathbb{Z}$ as follows: for $(\gamma, m) \in \mathfrak{p}^{\infty} \times \mathbb{Z}$, define $f : \mathfrak{p}^{\infty} \times \mathbb{Z} \longrightarrow \mathfrak{p}_{\mathcal{G} \times_1 \mathbb{Z}}^{\infty}$ by

$$f(\gamma, m) := (e_1, m)(e_2, m+1)\ldots,$$

for $\gamma = e_1 e_2 \ldots$. It is straightforward to check that this defines an infinite path in $\mathcal{G} \times_1 \mathbb{Z}$. Define a shift $\sigma : \mathfrak{p}^{\infty} \longrightarrow \mathfrak{p}^{\infty}$ by the formula,

$$\sigma(\gamma) = e_2 e_3 \ldots, \gamma = e_1 e_2 e_3 \ldots \in \mathfrak{p}^{\infty},$$

and define another shift $\widetilde{\sigma} : \mathfrak{p}^{\infty} \times \mathbb{Z} \longrightarrow \mathfrak{p}^{\infty} \times \mathbb{Z}$ by the formula, $\widetilde{\sigma}(\gamma, n) = (\sigma(\gamma), n+1)$. Under this identification the groupoid model $\mathfrak{G}_{\mathcal{G}}$ for $\mathcal{G}$ is given

by the equation,

$$\mathfrak{G}_{\mathcal{G}}=\{(\gamma,m-l,\gamma'):\gamma,\gamma'\in\mathfrak{p}^{\infty},\sigma^{m}(\gamma)=\sigma^{l}(\gamma')\},$$

while the groupoid model $\mathfrak{G}_{\mathcal{G}\times_1\mathbb{Z}}$ for $\mathcal{G}\times_1\mathbb{Z}$ is given by the equation,

$$\mathfrak{G}_{\mathcal{G}\times_1\mathbb{Z}}=\{((\gamma,n),m-l,(\gamma',k)):\gamma,\gamma'\in\mathfrak{p}^{\infty},\widetilde{\sigma}^{m}(\gamma,n)=\widetilde{\sigma}^{l}(\gamma',k),n,k\in\mathbb{Z}\}.$$

Define a map $\phi:\mathfrak{G}_{\mathcal{G}}\times_1\mathbb{Z}\longrightarrow\mathfrak{G}_{\mathcal{G}\times_1\mathbb{Z}}$ as follows: for γ and $\gamma'\in\mathfrak{p}^{\infty}$ with $\sigma^{m}(\gamma)=\sigma^{l}(\gamma')$ and $n\in\mathbb{Z}$, set

$$\phi\left[((\gamma,m-l,\gamma'),n)\right]:=((\gamma,n),m-l,(\gamma',n+m-l)).$$

Note that

$$\begin{aligned}\widetilde{\sigma}^{m}(\gamma,n)&=(\sigma^{m}(\gamma),n+m)\\&=\left(\sigma^{l}(\gamma'),n+m\right)\\&=\left(\sigma^{l}(\gamma'),(n+m-l)+l\right)\\&=\widetilde{\sigma}^{l}(\gamma',n+m-l),\end{aligned}$$

hence $((\gamma,n),m-l,(\gamma',n+m-l))\in\mathfrak{G}_{\mathcal{G}\times_1\mathbb{Z}}$. So ϕ is well defined. The rest of the proof proceeds as in [4, Theorem 2.4]. ■

In order to show that $C^{*}(\mathcal{G})\rtimes_{\gamma}\mathbb{T}$ is AF, we need the following lemma.

Lemma 27 *Let $\mathcal{G}=\left(G^{0},\mathcal{G}^{1},r,s\right)$ be an ultragraph. Let $G=(\mathbb{Z},+)$ be the discrete group of the integers under addition and let $\varphi:\mathcal{G}^{1}\longrightarrow\mathbb{Z}$ be the function defined by $\varphi(e)=1$, for all $e\in\mathcal{G}^{1}$. Then the ultragraph C^{*}-algebra, $C^{*}(\mathcal{G}\times_1\mathbb{Z})$ is an AF-algebra.*

Proof. Observe that the ultragraph, $\mathcal{G}\times_1\mathbb{Z}$, has no loops. Thus by Theorem 4.1, in [11], $C^{*}(\mathcal{G}\times_1\mathbb{Z})$ is an AF-algebra. ■

Corollary 28 *Let $\mathcal{G}=\left(G^{0},\mathcal{G}^{1},r,s\right)$ be an ultragraph, with no singular vertices. Then the groupoid C^{*}-algebra, $C^{*}(\mathfrak{G}_{\mathcal{G}}\times_c\mathbb{Z})$ is an AF-algebra. Furthermore, $C^{*}(\mathfrak{G}_{\mathcal{G}}\times_c\mathbb{Z})$ is nuclear and hence $\mathfrak{G}_{\mathcal{G}}\times_c\mathbb{Z}$ is amenable.*

Proof. The proof follows from Theorem 26 and Lemma 27. ■

Theorem 29 *If $\mathcal{G}$ is an ultragraph with no singular vertices, then $C^{*}(\mathcal{G})\rtimes_{\gamma}\mathbb{T}$ is AF and the groupoid $\mathfrak{G}_{\mathcal{G}}$ is amenable.*

Proof. Fix $z\in\mathbb{T}$ and let $\alpha_z(f)(h)=z^{c(h)}f(h)$, for $f\in C_c(\mathfrak{G}_{\mathcal{G}})$, $h\in\mathfrak{G}_{\mathcal{G}}$. Then $\alpha_z\in\mathrm{Aut}C^{*}(\mathfrak{G}_{\mathcal{G}})$ and $\{\alpha_z\}_{z\in\mathbb{T}}$ is a strongly continuous action of $\mathbb{T}$ on $C^{*}(\mathfrak{G}_{\mathcal{G}})$ (see [8, Proposition 5.1, p.110]). So we can form the crossed product C^{*}-algebra $C^{*}(\mathfrak{G}_{\mathcal{G}})\rtimes_{\alpha}\mathbb{T}$, and by Theorem 5.7 in [8, p.118], we have $C^{*}(\mathfrak{G}_{\mathcal{G}})\rtimes_{\alpha}\mathbb{T}\simeq C^{*}(\mathfrak{G}_{\mathcal{G}}\times_c\mathbb{Z})$. Recall the gauge action γ of $\mathbb{T}$ on $C^{*}(\mathcal{G})$. Since $C^{*}(\mathcal{G})$ is defined to be the universal C^{*}-algebra generated by the s_e, p_A's subject to the

relations in Definition 2 and the gauge action on $C^*(\mathcal{G})$ preserves these relations (and so does α via $s_e \longrightarrow 1_{\mathcal{A}'((e,r(e)),r(e))}$ and $p_A \longrightarrow 1_{\mathcal{A}'(A,A)}$), we have

$$\begin{aligned} C^*(\mathcal{G}) \rtimes_\gamma \mathbb{T} &\simeq C^*(\mathfrak{G}_\mathcal{G}) \rtimes_\alpha \mathbb{T} \\ &\simeq C^*(\mathfrak{G}_\mathcal{G} \times_c \mathbb{Z}) \\ &\simeq C^*(\mathfrak{G}_{\mathcal{G}\times_1\mathbb{Z}}) \simeq C^*(\mathcal{G} \times_1 \mathbb{Z}). \end{aligned}$$

Thus $C^*(\mathcal{G}) \rtimes_\gamma \mathbb{T}$ is an AF-algebra. Corollary 28 implies that $\mathfrak{G}_\mathcal{G} \times_c \mathbb{Z}$ is amenable. Since $\mathbb{Z}$ is amenable we may apply [8, Proposition II.3.8] to deduce that $\mathfrak{G}_\mathcal{G}$ is amenable. ■

As we mentioned earlier, we can extend our result to general ultragraphs using desingularization.

Theorem 30 *All ultragraph groupoids are amenable.*

Proof. Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph, and let $\mathcal{F}$ be a desingularization of $\mathcal{G}$. Then $\mathcal{F}$ is an ultragraph with no singular vertices. Thus the groupoid $\mathfrak{G}_\mathcal{F}$ is amenable. But then we have $C^*(\mathfrak{G}_\mathcal{G}) \simeq C^*(\mathcal{G})$ and $C^*(\mathcal{F}) \simeq C^*(\mathfrak{G}_\mathcal{F})$. By Theorem 6.6 in [10] $C^*(\mathcal{F})$ is strongly Morita equivalent to $C^*(\mathcal{G})$. Thus $C^*(\mathfrak{G}_\mathcal{G})$ is strongly Morita equivalent to $C^*(\mathfrak{G}_\mathcal{F})$. Therefore $\mathfrak{G}_\mathcal{G}$ is amenable. ■

4 THE SIMPLICITY OF $C^*(\mathcal{G})$

In this section we will use the groupoid $\mathfrak{G}_\mathcal{G}$ to obtain conditions sufficient for $C^*(\mathcal{G})$ to be simple. The result obtained is effectively due to Mark Tomforde who also proved the converse (see [11, Theorem 3.11]). It seems likely that the approach to the converse can be adapted to apply within the groupoid context of the present paper.

For a vertex $v \in G^0$, *a loop based at* v is a finite path $\alpha = e_1 \dots e_n$ in $\mathcal{G}$, such that $s(\alpha) = v$, $v \in r(\alpha)$ and $v \neq s(e_i)$ for all $1 < i \leq n$. When α is a loop based at v, we say that v *hosts the loop* α. A loop based at v may pass through other vertices $w \neq v$ more than once but no edge other than e_1 may have source v. The ultragraph $\mathcal{G}$ is said to satisfy *condition (K)* if every $v \in G^0$ which hosts a loop hosts at least two distinct loops. (See [2, Defintion 7.1, p.17,18].)

Recall that a locally compact groupoid G is *essentially principal* ([8, p.100]) if for all nonempty closed invariant subset F of its unit space, $G^{(0)}$ the set, $\{x \in F : x \text{ has trivial isotropy}\}$, is dense in F. We now show that for a general ultragraph $\mathcal{G}$, the groupoid $\mathfrak{G}_\mathcal{G}$ is essentially principal if and only if $\mathcal{G}$ satisfies condition (K).

Theorem 31 *If $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ is an ultragraph, the r-discrete groupoid $\mathfrak{G}_\mathcal{G}$ is essentially principal if and only if $\mathcal{G}$ satisfies condition (K).*

Proof. Suppose that $\mathcal{G}$ satisfies condition (K). Every ultrapath in $\mathfrak{p}$ has trivial isotropy. So we just need to consider the infinite paths. Let F be a nonempty

closed invariant subset of $\mathfrak{p}^{\infty}$. We have to show that the set of points in F with trivial isotropy is dense in F. So fix any $\chi \in F$, and fix a basic open neighborhood $D_{(x,x)} \cap F$ of χ, $(x \in \mathfrak{p}, |x| \geq 1)$. Note that χ must have the form $x \cdot \gamma$ where $\gamma \in \mathfrak{p}^{\infty}$ and $s(\gamma) \in r(x)$. (See Lemma 14.) If every vertex through which γ passes hosts no loop, then γ must pass through each of them exactly once. Thus the triple, $(x \cdot \gamma, k, x \cdot \gamma)$ can belong to $\mathfrak{G}_{\mathcal{G}}$ only if $k = 0$, and χ itself has trivial isotropy. So we may assume that the infinite path γ passes through some vertex v hosting a loop, say $\gamma = \beta \cdot \gamma'$, $\gamma' \in \mathfrak{p}^{\infty}$, with $s(\gamma') = v$. Let μ and ν be distinct loops based at v, and define paths $\gamma_n \in \mathfrak{p}^{\infty}$ by

$$\gamma_n := x \cdot \beta\mu\nu\mu\mu\nu\nu \cdots \overbrace{\mu \cdots \mu}^{n}\overbrace{\nu \cdots \nu}^{n} \cdot \gamma'.$$

Observe that each triple, $(\gamma_n, k_n, x \cdot \beta \cdot \gamma')$, belongs to $\mathfrak{G}_{\mathcal{G}}$, (with a substantial lag k_n), and since F is invariant, each γ_n lies in F. The sequence γ_n converges to the infinite path,

$$x \cdot \beta\mu\nu\mu\mu\nu\nu \cdots \overbrace{\mu \cdots \mu}^{n}\overbrace{\nu \cdots \nu}^{n} \cdots,$$

which has trivial isotropy and belongs to $D_{(x,x)} \cap F$ because F is closed. So we have approximated χ by a point with trivial isotropy. Thus the groupoid $\mathfrak{G}_{\mathcal{G}}$ is essentially principal.

Suppose conversely that $\mathfrak{G}_{\mathcal{G}}$ is essentially principal and suppose that $v \in G^0$ is a vertex hosting exactly one loop $\alpha = e_1 \ldots e_n$. Consider the set,

$$C = \{\gamma = \gamma_1\gamma_2 \ldots \in \mathfrak{p}^{\infty} : s(\gamma_i) \geq v \text{ for all } i \geq 1\}.$$

Note that the infinite path $\gamma = \alpha\alpha \ldots$ belongs to C. Let $F = \overline{C}$, the closure of C in X. We show F is invariant subset of X. So take any $h = (\chi, k, \chi') \in \mathfrak{G}_{\mathcal{G}}$ and suppose that $\chi \in F$. Then there is a sequence of infinite paths, $\gamma_n \in C$ such that $\gamma_n \longrightarrow \chi$. Either $|\chi| = \infty$ or $|\chi| < \infty$. Suppose first that $|\chi| = \infty$. Then for some $x, y \in \mathfrak{p}$, and $\mu \in \mathfrak{p}^{\infty}$, we have $\chi = x \cdot \mu$ and $\chi' = y \cdot \mu$. Then eventually, every $\gamma_n = x \cdot \mu_n$, where μ_n is a sequence in $\mathfrak{p}^{\infty}$ such that $s(\mu_n) \in r(x)$, and so $y \cdot \mu_n \longrightarrow y \cdot \mu$ eventually. Next we show that $y \cdot \mu_n \in C$ eventually. It will follow that $\chi' = y \cdot \mu \in F$. For this end, if $|y| = 0$, then since $x \cdot \mu_n \in C$ eventually, μ_n must be in C eventually. But since $|y| = 0$, the equation, $y \cdot \mu_n = \mu_n$, holds. Thus $y \cdot \mu_n \in C$ eventually. Otherwise let $y = (\beta, B)$ for some finite path β with positive length. Set $\beta = \beta_1 \ldots \beta_{|\beta|}$, and, $\mu_n = \mu_{n1}\mu_{n2} \ldots$. Fix any $i \in \{1, \ldots, |\beta|\}$. Let η be the finite path starting at $s(\beta_i)$ with range, $r(\mu_{n_1})$, that is, $\eta = \beta_i\beta_{i+1} \ldots \beta_{|\beta|}\mu_{n_1}$. Since $x \cdot \mu_n \in C$ eventually, we may choose a finite path δ with $s(\delta) = s(\mu_{n_2})$ and $v \in r(\delta)$. So set $\theta = \eta \cdot \delta$, which belongs to $\mathcal{G}^*$. Further, the equations, $s(\theta) = s(\eta) = s(\beta_i)$ and $v \in r(\delta) = r(\theta)$, hold. Hence $s(\beta_i) \geq v$ for each $i \in \{1, \ldots |\beta|\}$, and since each $s(\mu_{n_j}) \geq v$, we may conclude that $y \cdot \mu_n \in C$ eventually. So $y \cdot \mu \in F$ and $\chi' \in F$.

If $|\chi| < \infty$, then $r(\chi) = r(\chi')$, and a similar argument gives that $\chi' \in F$. We may use exactly the same argument to show that, if $\chi' \in F$ then $\chi \in F$. Therefore F is invariant.

We will contradict the assumption on the vertex v by showing that if $\mu \in F$ and $s(\mu) = v$, then $\mu = \gamma$. (For then, any sequence in F converging to γ eventually will not have trivial isotropy.) We can suppose that $\mu \in C$, since each $\mu' \in F$ of finite length and with $s(\mu') = v$ is the limit of sequence of such an infinite path μ.

Set $\mu = e_1' e_2' \ldots$ and for each n, let $y = e_1' \ldots e_n'$. Since $\mu \in C$, we have $s(e_n') \geq v$. So there is a finite path β such that $s(\beta) = s(e_n')$ and $v \in r(\beta)$. Note that $\alpha' := e_1' \ldots e_n' \cdot \beta = y \cdot \beta$ is another loop based at v. Since α is the only loop based at v, α' is of the form $\alpha\alpha \ldots \alpha$. It follows that every initial segment of μ is an initial segment of γ. Therefore $\mu = \gamma$ and hence the ultragraph $\mathcal{G}$ must satisfy condition (K). ■

The following corollary is an immediate consequence of Theorem 31 and that fact that ultragraph groupoids are amenable.

Theorem 32 *If $\mathcal{G}$ is an ultragraph satisfying condition (K), then the ideals in $C^*(\mathcal{G})$ are in bijective correspondence with the open invariant subsets of the unit space of $\mathfrak{G}_{\mathcal{G}}$. In particular, $C^*(\mathcal{G})$ is simple if and only if $\mathfrak{G}_{\mathcal{G}}$ is minimal.*

Recall that to say a groupoid is minimal is simply to say that the only invariant open subsets of its unit space are the empty set and the entire unit space.

Proof. Theorem 31 shows that $\mathfrak{G}_{\mathcal{G}}$ is essentially principal when $\mathcal{G}$ satisfies condition (K). On the other hand, Theorem 28 shows that all ultragraph groupoids are amenable and, therefore, that $C^*(\mathcal{G}) \simeq C^*(\mathfrak{G}_{\mathcal{G}}) = C^*_{\mathrm{red}}(\mathfrak{G}_{\mathcal{G}})$. Thus, the result follows from [8, Proposition 2.4.6]. ■

We conclude with a groupoid approach to the sufficiency part of Theorem 3.11 in [11], which gives necessary and sufficient conditions for an ultragraph C^*-algebra to be simple. First we introduce the following definition.

Definition 33 *Let $\mathcal{G} = \left(G^0, \mathcal{G}^1, r, s\right)$ be an ultragraph and v be a vertex. Let A be a set in $\mathcal{G}^0$ and let α be a finite path in $\mathcal{G}^*$. Then we write $v \longrightarrow_\alpha A$, to mean that $s(\alpha) = v$ and $A \subseteq r(\alpha)$. Roughly speaking the vertex v reaches the set A via one path α. Compare with [11, p.909].*

Theorem 34 *Let $\mathcal{G} = \left(G^0, \mathcal{G}^1, r, s\right)$ be an ultragraph satisfying condition (K). Then the ultragraph C^*-algebra $C^*(\mathcal{G})$ is simple if the following two conditions hold.*

(*1*) *$\mathcal{G}$ is cofinal ([11]) in the sense that given a vertex v and an infinite path $\gamma \in \mathfrak{p}^\infty$, there exists an n such that $v \geq s(\gamma_n)$; and*

(*2*) *if $A \in \mathcal{G}^0$ emits infinitely many edges in $\mathcal{G}^1$, then for every $v \in G^0$ there exists a finite path $\alpha \in \mathcal{G}^*$ such that $v \longrightarrow_\alpha A$.*

Proof. Suppose that $\mathcal{G}$ satisfies condition (K) and the two conditions *(1)* and *(2)*. By the preceding comments, we just need to show that $\mathfrak{G}_{\mathcal{G}}$ is minimal. Let $U \neq \emptyset$ be an open invariant subset of $\mathfrak{G}_{\mathcal{G}}^{(0)} = Y_\infty \cup \mathfrak{p}^\infty$. Since $\mathfrak{p}^\infty$ is dense in

$Y_\infty \cup \mathfrak{p}^\infty$, the inequality, $U \cap \mathfrak{p}^\infty \neq \emptyset$, holds. By considering a neighborhood in $\mathfrak{G}_{\mathcal{G}}^{(0)}$ of some $\gamma \in U \cap \mathfrak{p}^\infty$ we have $D_{((\alpha,A),(\alpha,A))} \cap \mathfrak{G}_{\mathcal{G}}^{(0)} \subset U$ for some $\alpha \in \mathcal{G}^*$ and some $A \in \mathcal{G}^0$ with $A \subseteq r(\alpha)$. (See Lemma 14.) Pick a vertex $v \in A$. Take any $\gamma = e_1 e_2 \ldots \in \mathfrak{p}^\infty$. Then by *(1)*, $v \geq s(e_i)$ for some $i \in \mathbb{N}$. Then there is a finite path $\beta \in \mathcal{G}^*$ such that $s(\beta) = v$ and $s(e_i) \in r(\beta)$. Notice that the triple, $(e_1 \ldots e_i e_{i+1} \ldots, |e_1 \ldots e_i| - |\alpha\beta e_i|, \alpha\beta e_i e_{i+1} \ldots)$, belongs to $\mathfrak{G}_{\mathcal{G}}$. Since $\alpha\beta e_i e_{i+1} \ldots \in D_{((\alpha,A),(\alpha,A))} \cap \mathfrak{G}_{\mathcal{G}}^{(0)} \subset U$ and since U is invariant, we must have $\gamma \in U$. Next take any $y \in Y_\infty$. Then the range of y, $r(y)$, is an infinite emitter. Then by *(2)* $v \longrightarrow_{\alpha'} r(y)$ for some $\alpha' \in \mathcal{G}^*$. Thus $v = s(\alpha')$ and $r(y) \subseteq r(\alpha')$. Observe that, the ultrapath, $(\alpha\alpha', r(y))$, lies in Y_∞. It follows, then, that the triple, $((\alpha\alpha', r(y)), |\alpha\alpha'| - |y|, y)$, belongs to $\mathfrak{G}_{\mathcal{G}}$. (See Notation 1.1.) Since $(\alpha\alpha', r(y)) \in D_{((\alpha,A),(\alpha,A))} \cap \mathfrak{G}_{\mathcal{G}}^{(0)} \subset U$ and since U is invariant, we must have $y \in U$. Thus, the inclusion, $\mathfrak{G}_{\mathcal{G}}^{(0)} = Y_\infty \cup \mathfrak{p}^\infty \subseteq U$, holds, and hence $\mathfrak{G}_{\mathcal{G}}^{(0)} = U$. So $\mathfrak{G}_{\mathcal{G}}$ is minimal. ■

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